

ON FUNCTIONS WHOSE SYMMETRIC PART OF GRADIENT AGREE AND A GENERALIZATION OF RESHETNYAK'S COMPACTNESS THEOREM

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ABSTRACT. We consider the following question: Given a connected open domain $\Omega \subset \mathbb{R}^n$, suppose $u, v : \Omega \rightarrow \mathbb{R}^n$ with $\det(\nabla u) > 0$, $\det(\nabla v) > 0$ a.e. are such that $\nabla u^T(x)\nabla u(x) = \nabla v(x)^T\nabla v(x)$ a.e., does this imply a global relation of the form $\nabla v(x) = R\nabla u(x)$ a.e. in Ω where $R \in SO(n)$? If u, v are C^1 it is an exercise to see this true, if $u, v \in W^{1,1}$ we show this is false. In Theorem 1 we prove this question has a positive answer if $v \in W^{1,1}$ and $u \in W^{1,n}$ is a mapping of L^p integrable dilatation for $p > n - 1$. These conditions are sharp in two dimensions and this result represents a generalization of the corollary to Liouville's theorem that states that the differential inclusion $\nabla u \in SO(n)$ can only be satisfied by an affine mapping.

Liouville's corollary for rotations has been generalized by Reshetnyak who proved convergence of gradients to a fixed rotation for any weakly converging sequence $v_k \in W^{1,1}$ for which

$$\int_{\Omega} \text{dist}(\nabla v_k, SO(n)) dz \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let $S(\cdot)$ denote the (multiplicative) symmetric part of a matrix. In Theorem 3 we prove an analogous result to Theorem 1 for any pair of weakly converging sequences $v_k \in W^{1,p}$ and $u_k \in W^{1, \frac{p(n-1)}{p-1}}$ (where $p \in [1, n]$ and the sequence (u_k) has its dilatation pointwise bounded above by an L^r integrable function, $r > n - 1$) that satisfy $\int_{\Omega} |S(\nabla u_k) - S(\nabla v_k)|^p dz \rightarrow 0$ as $k \rightarrow \infty$ and for which the sign of the $\det(\nabla v_k)$ tends to 1 in L^1 . This result contains Reshetnyak's theorem as the special case $(u_k) \equiv Id$, $p = 1$.

Rigidity of differential inclusions under minimal regularity has been a much studied topic. Probably the best known problem of this type is the study of the validity of Liouville's theorem [Lio 50] characterizing functions u that satisfy the differential inclusion,

$$\nabla u \in CO_+(n) := \{\lambda R : \lambda > 0, R \in SO(n)\}.$$

Liouville's original theorem was proved for C^4 mappings in \mathbb{R}^3 . This was later generalized by Gehring [Ge 62], Reshetnyak [Re 67], Bojarski and Iwaniec [Bo-Iw 82], Iwaniec and Martin [Iw-Ma 93], [Mu-Sv-Ya 99]. In even dimensions the minimal regularity for Liouville theorem to hold is $u \in W^{1, \frac{n}{2}}(\Omega)$ (examples show no better result is possible). In odd dimensions the optimal regularity is unknown but is conjectured to be $\frac{n}{2}$.

A corollary to Liouville's theorem is that functions whose gradient belongs to $SO(n)$ are affine. Note that if $u \in W^{1,1}(\Omega)$ and $\nabla u \in SO(n)$ then $\text{div}(\nabla u) = \text{div}(\text{cof}(\nabla u)) = 0$. Thus every co-ordinate function of u weakly satisfies Laplace's equation and hence by Weyl's lemma is C^∞ , thus rigidity for this differential inclusion follows by elementary means. However under a much weaker assumptions that $u \in SBV(\Omega)$, $\nabla u \in SO(n)$ a 'piecewise' rigidity result has been established in [Ch-Gi-Po 07]. Note that the rigidity of the differential inclusion $\nabla u \in SO(n)$ for $\nabla u \in W^{1,1}$ follows as a highly special case of the following 'first guess' conjecture.

'First guess' conjecture. Let $\Omega \subset \mathbb{R}^n$ be a connected open domain, let $u, v \in W^{1,1}(\Omega)$ and $\det(\nabla u) > 0$,

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$\det(\nabla v) > 0$ for a.e. with

$$\nabla u(x)^T \nabla u(x) = \nabla v(x)^T \nabla v(x) \text{ for a.e. } x \in \Omega$$

then there exists $R \in SO(n)$ such that $\nabla v = R \nabla u$ a.e.

As we will show in Example 1, Section 4, this conjecture is false. One of the principle aims of this paper will be to establish sufficient regularity assumptions required for the differential equality $\nabla u^T \nabla u = \nabla v^T \nabla v$ to imply $\nabla u = R \nabla v$ for some $R \in SO(n)$. If $u, v \in C^1$ this property would be easy to prove, for $W^{1,1}$ it is not true. In Theorem 1 below we establish the validity of this conjecture with respect to a condition that is sharp in two dimensions.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a connected open domain, let $v \in W^{1,1}(\Omega : \mathbb{R}^n)$ and $u \in W^{1,n}(\Omega : \mathbb{R}^n)$, $\det(\nabla v) > 0$, $\det(\nabla u) > 0$ a.e. and $\|\nabla u(x)\|^n \leq K(x) \det(\nabla u(x))$ for $K \in L^{P_n}$ where*

$$P_n := \begin{cases} 1 & \text{for } n = 2 \\ > n - 1 & \text{for } n \geq 3 \end{cases} . \quad (1)$$

Suppose

$$\nabla u(x)^T \nabla u(x) = \nabla v(x)^T \nabla v(x) \text{ for a.e. } x \in \Omega \quad (2)$$

then there exists $R \in SO(n)$

$$\nabla v(x) = R \nabla u(x) \text{ for a.e. } x \in \Omega. \quad (3)$$

Much interest in the differential inclusion $\nabla u \in SO(n)$ comes from recent powerful generalization of the corollary to Liouville's theorem that has been established in Theorem 3.1 [Fr-Ja-Mu 02]. Specifically the L^2 distance of the gradient of a function away from a *fixed* rotation was shown to be bounded by a constant multiple of the L^2 distance of the gradient away from the *set* of rotations¹. Previously strong partial results controlling the function (rather than the gradient) have been established by John [Jo 61], Kohn [Ko 82]. There has been much work generalizing Theorem 3.1 of [Fr-Ja-Mu 02], for example [Ch-Mu 03], [Fa-Zh 05], [Lo 05], [Co-Sc 06], [Je-Lor 08], [Cm-Co 10]. Part of the motivation for Theorem 1 is to open a new direction of generalization of Theorem 3.1 of [Fr-Ja-Mu 02]. For a quantitative version of Theorem 1 in two dimensions using quite different methods see our companion paper [Lo 10].

Prior to the advances made in [Fr-Ja-Mu 02] the most general result generalizing Liouville's theorem for mappings with gradient in the space of rotations that gave some control of the gradient was due to Reshetnyak [Re 67], we state his theorem for bounded connected domains.

Theorem 2 (Reshetnyak 1967). *Let Ω be an open connected and bounded set. If v_k converges weakly in $W^{1,1}(\Omega : \mathbb{R}^n)$ and*

$$\int_{\Omega} \text{dist}(\nabla v_k, SO(n)) dx \rightarrow 0 \text{ as } k \rightarrow \infty \quad (4)$$

then ∇v_k converges strongly in L^1 to a single matrix in $SO(n)$.

Reshetnyak's Theorem is an example of a result in the more general theory of stability of approximate differential inclusions. Specifically the study of what conditions a set of matrices K must have in order for $\int_{\Omega} \text{dist}(\nabla v_k, K) dx \rightarrow 0$ to imply $\{\nabla v_k\}$ is compact in $L^1(\Omega)$ for a uniformly bounded Lipschitz sequence, [Mu 96], [Ta 79], [Mu-Sv-Ya 99]. The study of these sets of matrices is closely connected to the theory of quasiconvexity in the calculus of variations [Ba 77], [Mo 52] and was largely motivated by the work of Ball and James [Ba-Ja 87], [Ba-Ja 92], Chipot and Kinderlehrer [Ch-Ki 88] on variational models of crystal microstructure. The main result of our paper is a generalization of Reshetnyak's theorem, setting $u_k \equiv Id$, $p = 1$ in Theorem 3 below we recover

¹A straightforward adaption of the proof of Theorem 3.1, [Fr-Ja-Mu 02] establishes the same result for L^p control, where $p > 1$.

Theorem 2. In the statement of the theorem and from this point on we let $S(\cdot)$ denote the (multiplicative) symmetric part of a matrix. Let $\text{sgn}(\cdot)$ denote the sign of a number, i.e. $\text{sgn}(x) = \frac{x}{|x|}$ for $x > 0$ and -1 otherwise.

Theorem 3. Let $\Omega \subset \mathbb{R}^n$ be open and connected and bounded. Let $p \in [1, n]$, $q = \frac{p(n-1)}{p-1}$. Suppose $v_k \in W^{1,p}(\Omega : \mathbb{R}^n)$ converges weakly in $W^{1,p}$ with $\text{sgn}(\det(\nabla v_k)) \xrightarrow{L^1} 1$ and $u_k \in W^{1,q}(\Omega : \mathbb{R}^n)$ converges weakly in $W^{1,q}$, satisfies $\det(\nabla u_k) > 0$ a.e. and $\|\nabla u_k\|^n \leq K \det(\nabla u_k)$ for all k where $K \in L^{P_n}$ and P_n satisfies (1). If

$$\int_{\Omega} |S(\nabla v_k) - S(\nabla u_k)|^p dx \rightarrow 0 \text{ as } k \rightarrow \infty \quad (5)$$

then there exists $R \in SO(n)$ such that

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\nabla v_k - R \nabla u_k| dx = 0. \quad (6)$$

For $p = n$, Theorem 3 provides a sharp answer to the question, what is the hypothesis necessary such that two weakly converging sequences $\nabla u_k, \nabla v_k \in W^{1,n}$ with $\int_{\Omega} |S(\nabla u_k) - S(\nabla v_k)|^n dz \rightarrow 0$ have the property that there must exists $R \in SO(n)$ so that $\lim_{k \rightarrow \infty} \int_{\Omega} |\nabla v_k - R \nabla u_k| dx = 0$.

By taking $u_k \equiv u$, $v_k \equiv v$ we see Theorem 3 generalizes Theorem 1. Example 1 from Section 4 shows the necessity (and sharpness in two dimensions) of the condition on (u_k) . The condition on (v_k) can also easily be seen to be necessary, for example by considering $u_k \equiv Id$, $v_k \equiv v$ where v is a non affine Lipschitz mapping with its gradient in the set $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

Another direction of generalization of Theorem 2 was proved by Müller, Sverak and Yan [Mu-Sv-Ya 99] who generalized Theorem 2 for a weakly converging sequence $u_k \in W^{1, \frac{n}{2}}$ where the set of rotations in (4) is replaced by the set of conformal matrices $CO_+(n)$.

As should seem likely from the assumptions of Theorems 3 we will be using the powerful results established by Iwaniec and Sverak [Iw-Sv 93], Villamore and Manfredi [Ma-Vi 98], Koskela and Heinonen [He-Ko 93] on functions of *integrable dilatation*. These are functions u for which $L(x) := \frac{\|\nabla u(x)\|^n}{\det(\nabla u(x))}$ is a positive L^p integrable function, if L is merely positive and finite a.e. we say u is a mapping of *finite dilatation*. Following [Iw-Sv 93] there has been a well known conjecture that if u is a mapping of finite dilatation where $L \in L^{n-1}$ then u is open and discrete. The best result to date has been established by Villamore and Manfredi [Ma-Vi 98] whose proved the conjecture for functions that satisfy $L \in L^p$ for $p > n - 1$. If the conjecture was true for $L \in L^{n-1}$ then Theorem 3 would hold for $K \in L^{n-1}$. It is however not clear for $n \geq 3$ if this is the optimal result.

On sharpness. The counter example to the ‘first guess’ conjecture that we construct in Section 4 works by squeezing down the center of the square to a point so that the interior of the image is disjoint. All known counter examples in higher dimension work in a similar way. If it turned out that L^p (for $p > n - 1$) integrability of the dilatation $\frac{\|\nabla u\|^n}{\det(\nabla u)}$ was a sharp condition to prevent this, it would suggest this condition is sharp for Theorem 1 and Theorem 3. With this in mind, in Section 5 we consider mappings from the cylinder $B_1(0) \times [0, 1]$ such that $u(B_1(0) \times \{0\})$ consists of a point. If it could be shown such mappings exists with $\int_{B_1(0) \times [0, 1]} \left(\frac{\|\nabla u\|^3}{\det(\nabla u)} \right)^p dz < \infty$ for $p < 2$ and $p \sim 2$ then Theorems 1, 3 would be sharp. However in Proposition 1 it is shown that any radial mapping u of the cylinder that squeezes one end to a point but for which each co-ordinate function is a product of functions in cylindrical polar co-ordinates that are monotonic and convex or concave, then $\int_{B_1(0) \times [0, 1]} \frac{\|\nabla u\|^3}{\det(\nabla u)} dz = \infty$. Our guess is that Theorem 1, Theorem 3 are not sharp for $n \geq 3$ and we suspect these theorems holds true for functions of integrable dilatation.

Connections with Stylov decomposition and future directions. It is worth noting that in two dimensions the validity of ‘first guess conjecture’ is a special case of a more general question.

First some background, given $w : \Omega \rightarrow \mathbb{R}^2$, $w(x, y) = (u(x, y), v(x, y))$, for $z = x + iy$ let $\tilde{w}(z) = u(x, y) + iv(x, y)$. Note $\frac{\partial \tilde{w}}{\partial \bar{z}}(z) = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})\tilde{w} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$. And $\frac{\partial \tilde{w}}{\partial z}(z) = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})\tilde{w} = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y)$. Now identifying complex numbers with conformal matrices in the standard way $[x + iy]_M = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ we have that $\nabla w(x, y) = \frac{1}{2} \left[\frac{\partial \tilde{w}}{\partial \bar{z}}(z) \right]_M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \left[\frac{\partial \tilde{w}}{\partial z}(z) \right]_M$. The Beltrami coefficient $\mu(z)$ of \tilde{w} is defined by $\frac{\partial \tilde{w}}{\partial \bar{z}}(z) = \mu(z) \frac{\partial \tilde{w}}{\partial z}(z)$, so $\mu(z)$ relates the conformal part of ∇w to the reflection of the anticonformal part of ∇w . Note that if we let L be an affine map with gradient $\lambda \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ then turning $L \circ w$ into a complex function we obtain $\lambda(\cos \theta + i \sin \theta)\tilde{w}$ and the Beltrami coefficient of this function is still $\mu(z)$. In other words the Beltrami coefficient does not notice changes in gradient made by scalar multiplication or by rotation. It is also not hard to see that if matrices A, B have identical Beltrami coefficient then $AB^{-1} \in CO_+(n)$ and thus Beltrami coefficient has two components and ‘encodes’ the geometry of *how* a matrix deforms a ball but does not encode any information about the rotation or the size. The symmetric part of the gradient has three components and describes both the geometry and the size. It should there for not be a surprise that given matrices $A, B \in \mathbb{R}^{2 \times 2}$, if $S(A) = S(B)$ then the Beltrami coefficient also agree. There exists a general factorization result known as ‘Stylov’ factorization; specifically for mappings u_1, u_2 of finite dilatation and whose Beltrami coefficients agree where u_1 is a homeomorphism, there exists holomorphic ϕ such that $u_1 = \phi \circ u_2$ (see Theorem 20.4.19 [As-Iw-Ma 10]). If in addition we know that the $S(\nabla u_1) = S(\nabla u_2)$ this implies $|\nabla \phi| \equiv 1$ and therefor ϕ is a rotation². For higher dimensions there is no ‘Stylov’ decomposition and not only are Theorems 1, 3 about non invertible mappings, the methods we use to establish them are of very different. It is worth noting however that the nature of the factorization is to relate by a conformal mapping any two mappings whose gradients pointwise deform the ball with the same geometry, ignoring size and rotation. In higher dimensions given matrix $A \in \mathbb{R}^{n \times n}$ if we consider $\frac{S(A)}{|S(A)|}$ this matrix encodes geometry ignoring size and rotation, so we could consider two functions u, v with the property that $\frac{S(\nabla u(x))}{|S(\nabla u(x))|} = \frac{S(\nabla v(x))}{|S(\nabla v(x))|}$ for a.e. $x \in \Omega$ and ask if these two functions are related by a conformal mapping. We make the following conjecture;

Conjecture 1. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open connected domain and $n \geq 3$. Given $u \in W^{1,n}(\Omega)$, $v \in W^{1,1}(\Omega)$ where $\det(\nabla u) > 0$, $\det(\nabla v) > 0$ a.e. and u satisfies $\int_{\Omega} \left(\frac{\|\nabla u\|^n}{\det(\nabla u)} \right)^p dz < \infty$ for some $p > n - 1$ and

$$\frac{S(\nabla u(z))}{|S(\nabla u(z))|} = \frac{S(\nabla v(z))}{|S(\nabla v(z))|} \text{ for a.e. } z \in \Omega \quad (7)$$

then there exists a Mobius transformation Φ such that $v = \Phi \circ u$.

Given that in two dimensions $\frac{S(A)}{|S(A)|} = \frac{S(B)}{|S(B)|}$ is equivalent to the Beltrami coefficients of A and B being equal Conjecture 1 would be a generalization to ‘Stylov’ factorization to $n \geq 3$, note however Conjecture 1 is not true in two dimensions (without the assumption of invertibility) as can easily be seen by the complex functions z^2, z^3 . One of the main tools we used to prove Theorems 1, 3 is the quantitative Liouville theorem for rotations of Friesecke, Muller and James [Fr-Ja-Mu 02]. In order to prove Conjecture 7 what would be required is a quantitative Liouville theorem for conformal matrices. A weakly quantitative result along these lines has been proved by Reshetnyak [Re 82], and a much stronger quantitative theorem been proved by Faraco and Zhong [Fa-Zh 05]

²Using this and some methods of this paper a short proof of Theorem 1 in two dimensions can be given.

for mappings whose gradient lies in a compact subset of $CO_+(n)$ that excludes 0. Using these theorems and the methods of this paper we plan to establish Conjecture 1 in a forthcoming work.

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1. PROOF SKETCH

1.1. Sketch of Theorem 1. We will begin by sketching the proof in the simplest case for smooth globally invertible u and progressively show how the assumptions can be weakened till we arrive at hypothesis of Theorem 1.

So first we have C^1 functions u, v where u is globally invertible. Recall for matrix $A \in \mathbb{R}^{n \times n}$ we let $S(A) = \sqrt{A^T A}$ be the symmetric part of A and by polar decomposition we have $A = R(A)S(A)$ for some $R(A) \in SO(n)$. Form $w(z) = v(u^{-1}(z))$ and note that

$$\begin{aligned} \nabla w(z) &= \nabla v(u^{-1}(x))(\nabla u(u^{-1}(x)))^{-1} \\ &= R(\nabla v(u^{-1}(x))) \left(R(\nabla u(u^{-1}(x))) \right)^{-1} \in SO(n) \end{aligned}$$

by the Liouville’s theorem it’s clear there exists $R \in SO(n)$ such that $\nabla w(z) = R$ for all $z \in \Omega$. Thus

$$\nabla v = R \nabla u \text{ on } \Omega. \quad (8)$$

and result is established.

Now it can easily be seen that global invertibility is more than we need for this argument above to work, if we merely knew that for *every* $x \in \Omega$ there exists $r_x > 0$ such that $u|_{B_{r_x}(x)}$ is injective then we could use the same argument to show there exists $R_x \in SO(n)$ such that $R_x \nabla u = \nabla v$ on $B_{r_x}(x)$. Fix some x_0 and let

$$\mathcal{U} := \{x \in \Omega : R_{x_0} \nabla u(x) = \nabla v(x)\}.$$

For any $x \in \mathcal{U}$ we can show $R_x = R_{x_0}$ and thus \mathcal{U} is both open and closed. As Ω is connected it is clear that $\mathcal{U} = \Omega$. So if we merely have a set $\mathcal{I} \subset \Omega$ where $|\Omega \setminus \mathcal{I}| = 0$, \mathcal{I} is connected and u is locally injective on every point $x \in \mathcal{I}$ then the argument above will still carry through.

Now suppose $v, u \in W^{1,1}$ and u open and discrete then by a theorem of Chernavskii [Ch 64] we know that the set of points on which u fails to be locally injective (the so called ‘branch set’) which we denoted by B_u , is a set of topological dimension less than $n - 2$. Thus by Example VI 11 p93 [Wa 41] we know that $\Omega \setminus B_u$ is connected. However we are blocked from directly carrying out the previous argument by the fact that even if we knew $u^{-1} : u(B_{r_x}(x)) \rightarrow B_{r_x}(x)$ has Sobolev regularity it does not follow that $w = v \circ u^{-1}$ is defined or if it is defined to what extent some kind of chain rule holds for it. Therefore more regularity of u is required. If u was quasiregular then $u|_{B_{r_x}(x)}$ is quasiconformal and hence $u^{-1}|_{u(B_{r_x}(x))}$ is quasiconformal and so w would be a well defined Sobolev function and the chain rule holds for $v \circ u^{-1}$. Thus we could show $\nabla w \in SO(n)$ on $u(B_{r_x}(x))$ and the argument could be completed to establish $R \nabla u = \nabla v$ on Ω .

Now from the other direction let us consider how Theorem 1 could fail, take the map $P : Q_1(0) \rightarrow \mathbb{R}^2$ defined by $P(x, y) = (x, xy)$ for $x > 0$ and $P(x, y) = (x, -xy)$ for $x < 0$. So this map takes the unit square and squeezes the center down to form a bow tie. If we take another mapping H that leaves the left hand side of the bow tie alone and rotates down the right hand side. Then comparing $H \circ P$ and P we have that the symmetric part of the gradient of both of these functions agree almost everywhere, however we clearly have that there is no rotation R such that (8) holds true. For more details of this mapping see Example 1, Section 4. It is easy to see that the

dilatation $\frac{\|\nabla P(x,y)\|^2}{\det(\nabla P(x,y))} \sim x^{-1}$ and so is not integrable. On the other hand in two dimension from the work of Iwaniec, Sverak [Iw-Sv 93] we know mappings of integrable dilatation share many of the strong properties of quasiregular mappings. What is not clear for these mappings is if the chain rule holds for the composition $v \circ u^{-1}$, we do however at least know Sobolev regularly of u^{-1} by [He-Ko-Ma 06].

If we have a Lipschitz function f and a function $g \in W^{1,p}$ by considering the difference quotients of $f \circ g$ it is easy to see that $f \circ g \in W^{1,p}$. This does not mean that the chain rule holds, however in the case where $\det(\nabla g(x)) > 0$ for a.e. x we can apply the general BV chain rule of Ambrosio, Dal Maso [Am-Da 90]. Given this is the case a natural approach is for us to consider replacing v with a Lipschitz function \tilde{v} with the property that $\int |\nabla v - \nabla \tilde{v}|^p dx \approx 0$. Such a function can be found by the now standard truncation arguments via maximal functions of [Zh 92], [Ac-Fu 88]. The difficulty of this approach is that the composed function $\tilde{v} \circ u$ will not necessarily have its gradient in the set of rotations so the best we can hope for is an approximate differential inclusion

$$\int_{u(B_{r_x}(x))} d(\nabla(\tilde{v} \circ u^{-1}), SO(n)) dx \approx 0. \quad (9)$$

By use of the previously mentioned quantitative Liouville theorem of Frieesecke, Müller and James [Fr-Ja-Mu 02] we would then be able to conclude that there exists $R \in SO(n)$ such that

$$\int_{u(B_{r_x}(x))} |\nabla(\tilde{v} \circ u^{-1}) - R| dx \approx 0.$$

We have the following estimates

$$\begin{aligned} & \int_{u(B_{r_x}(x))} d(\nabla(\tilde{v}(u^{-1}(z))), SO(n)) dz \\ & \leq \int_{u(B_{r_x}(x))} |(\nabla \tilde{v}(u^{-1}(z)) - \nabla v(u^{-1}(z))) \nabla u(u^{-1}(z))^{-1}| dz \\ & \leq \int_{B_{r_x}(x)} |(\nabla \tilde{v}(y) - \nabla v(y)) \text{ADJ}(\nabla u(y))| dy. \end{aligned} \quad (10)$$

So in order to control this expression we need the appropriate integrability assumptions on ∇v , $\nabla \tilde{v}$ and ∇u . Since $v \in W^{1,p}(\Omega)$ so $\|v - \tilde{v}\|_{W^{1,p}} \approx 0$ and so by Holder's inequality we have

$$\int_{u(B_{r_x}(x))} d(\nabla \tilde{v}(u(z)), SO(2)) dz \leq \left(\int_{B_{r_x}(x)} |\nabla \tilde{v} - \nabla v|^p dz \right)^{\frac{1}{p}} \left(\int_{B_{r_x}(x)} |\nabla u|^{\frac{(n-1)p}{p-1}} dz \right)^{\frac{p-1}{p}} \approx 0.$$

So we can apply Frieesecke, Müller and James [Fr-Ja-Mu 02] and conclude that there exists $R_x \in SO(2)$ such that

$$\int_{u(B_{r_x}(x))} |\nabla(\tilde{v} \circ u^{-1}) - R_x| dz \approx 0.$$

Unwrapping this and taking the limit as $\tilde{v} \rightarrow v$ we have that $\nabla v = R_x \nabla u$ on $B_{r_x}(x)$ and we can complete the argument by showing this relation holds globally off the branch set of u .

1.2. Sketch of Theorem 3. The starting point for Theorem 3 is Theorem 1.4 of [Ge-Iw 99] that allows us to conclude that letting u denote the weak limit of u_k we have $\frac{\|\nabla u(z)\|^n}{\det(\nabla u(z))} \leq K(z)$ for a.e. $z \in \Omega$. Let v denote the weak limit of v_k . Since $u \in W^{1,n}(\Omega)$ and $v \in W^{1,1}(\Omega)$ for a.e. $x \in \Omega$ both u, v are approximately differentiable, hence from some $r_x > 0$ we have that $\frac{|u(z) - (u(x) + \nabla u(x)(z-x))|}{|z-x|} \approx 0$ and $\frac{|v(z) - (v(x) + \nabla v(x)(z-x))|}{|z-x|} \approx 0$ for all $z \in B_{r_x}(x)$. Now as $v_k \xrightarrow{L^1(\Omega)} v$ and $u_k \xrightarrow{L^1(\Omega)} u$ so for large enough k we have that v_k and u_k are very well approximated by the affine maps $W_x^v(z) := v(x) + \nabla v(x)(z-x)$ and $W_x^u(z) := u(x) + \nabla u(x)(z-x)$. Now for large

enough k we also know that $\int_{B_{r_x}(x)} |S(\nabla u_k) - S(\nabla v_k)| dz \approx 0$ and thus using Lemma 2 we have that there exists $R_x \in SO(n)$ such that

$$\int_{B_{r_x}(x)} |\nabla v_k - R_x \nabla u_k| dz \approx 0. \quad (11)$$

By Poincaré's inequality for some affine map L_x with $\nabla L_x = R_x$ we have $\int_{B_{r_x}(x)} |v_k - L_x \circ u_k| dz \approx 0$. Recall v_k and u_k are very well approximated by W_x^v and W_k^u thus it must follow that $\nabla v(x) = R_x \nabla u(x)$. This implies $S(\nabla v(x)) = S(\nabla u(x))$ for a.e. $x \in \Omega$ and hence we are in a position to apply Theorem 1. Thus there exists $R \in SO(n)$ such that $\nabla u(x) = R \nabla v(x)$ for a.e. $x \in \Omega$. Now again as v_k and u_k are L^∞ close to v, u by Poincaré's inequality from (11) we have that $R_x \approx R$. By covering Ω with a not too overlapping collection $\{B_{r_{x_1}}(x_1), B_{r_{x_2}}(x_2), \dots, B_{r_{x_q}}(x_q)\}$ we have that for each i , $R_{x_i} \approx R$ and so $\int_\Omega |\nabla v_k - R \nabla u_k| dx \approx 0$ for all large enough k .

Given the similarity between Lemma 2 and Lemma 1 it may seem curious that we need Lemma 2 at all. The reason is that the estimate in Lemma 1 gets control of $|\nabla u_k - r_k \nabla v_k|$ on a ball of radius $cr \exp\left(-\frac{A_k^r(x)}{\varepsilon^n}\right)$ where $A_k^r(x) = \int_{B_r(x)} |\nabla u_k|^n dx$. In order to obtain global control of $|\nabla u_k - R \nabla v_k|$ for some fixed $R \in SO(n)$ over the whole of some (large) subset $\Omega' \subset \Omega$ we would

$$\text{need a collection } \left\{ B_{cr_q \exp\left(-\frac{A_k^{r_q}(x_q)}{\varepsilon^n}\right)}(x_q) : x_q \in \Omega' \right\} \text{ for which}$$

$$\sum_q \mathbb{1}_{B_{cr_q \exp\left(-\frac{A_k^{r_q}(x_q)}{\varepsilon^n}\right)}(x_q)} \leq 5. \quad (12)$$

For this to work, (i.e. to be able to apply Lemma 1) we would need an estimate of the form $\sum_q A_k^{r_q}(x_q)(r_q)^n \leq c$ which would be available by equi-integrability of $|\nabla u_k|^n$ if $\{B_{r_q}(x_q) : q \in \mathbb{N}\}$ did not overlap by some fixed constant. However this completely fails to be a consequence of (12) and so no such estimate is available and more subtle arguments are needed to first establish Lemma 2 and get control of the functions in a ball of radius $\frac{r_q}{2}$ and then in the proof of Theorem 3 to carefully check the hypothesis of this Lemma 2 are satisfied.

2. PROOF OF THEOREM 1

Lemma 1. *Let $p \in [1, n]$, $q = \frac{p(n-1)}{p-1}$. Suppose $v \in W^{1,p}(B_r(x) : \mathbb{R}^n)$ and $u \in W^{1,q}(B_r(x) : \mathbb{R}^n)$ is a homeomorphism of integrable dilatation, i.e. there exists positive function $K \in L^1$ such that $\|\nabla u(z)\|^n \leq K(z) \det(\nabla u(z))$. Suppose for some constant $\varepsilon \in (0, 1)$*

$$B_{\varepsilon r}(u(x)) \subset u\left(B_{\frac{r}{4}}(x)\right) \quad (13)$$

and for $\varepsilon > 0$ such that $r^{-32n(n-1)} \leq \ln(2 + \varepsilon^{-\frac{1}{4}})$ we have

$$\int_{B_r(x)} |S(\nabla u) - S(\nabla v)|^p dz \leq \varepsilon \quad (14)$$

and

$$\int_{B_r(x)} |\operatorname{sgn}(\det(\nabla v)) - 1| dz \leq \varepsilon \quad (15)$$

then for positive constants $C_0 = C_0(n)$, $C_1 = C_1(n, \int_{B_r(x)} K dz)$ there exists $R \in SO(n)$ such that

$$\int_{B_{C_0 r \exp\left(-\frac{A_u^r}{\varepsilon^n}\right)}(x)} |\nabla u - R \nabla v| dz \leq C_1 A_u^r \exp\left(\frac{n A_u^r}{\varepsilon^n}\right) \left(\ln(2 + \varepsilon^{-\frac{1}{2}})\right)^{-\frac{1}{32n}}, \quad (16)$$

for

$$A_u^r := \int_{B_r(x)} |\nabla u|^q dz. \quad (17)$$

Proof of Lemma 1. First some notation, given subset S of \mathbb{R}^n or $\mathbb{R}^{n \times n}$ and $h > 0$ let

$$N_h(S) := \{X : \inf \{|X - Y| : Y \in S\} < h\}. \quad (18)$$

Note $q \geq p$ so by Holder, (14) and (17) implies that $\int_{B_{\frac{r}{2}}(x)} |\nabla v|^p dz \leq (A_u^r + 1)$. Define

$$M_\gamma := \left\{ z \in B_{\frac{r}{4}}(x) : \sup_{h \in (0, \frac{r}{4})} \int_{B_h(z)} |\nabla v|^p dz > \gamma \right\}.$$

We have $M_{\gamma_2} \subset M_{\gamma_1}$ for $\gamma_2 > \gamma_1$ and $|M_\gamma| \rightarrow 0$ as $\gamma \rightarrow 0$. Thus we can find $\lambda > 0$ large enough so that

$$\int_{\{z \in B_{\frac{r}{2}}(x) : |\nabla v(z)| > \lambda\}} |\nabla v|^p dz < \sqrt{\epsilon} r^n \quad (19)$$

and

$$\|\nabla v\|_{L^p(M_\lambda)} \leq c \epsilon^{\frac{1}{2p}} r^{\frac{n}{p}}. \quad (20)$$

Arguing as in Theorem 3, Section 6.6.3 [Ev-Ga 92] we have that

$$|M_\lambda| \leq c \lambda^{-p} \int_{\{z \in B_{\frac{r}{2}}(x) : |\nabla v(z)| > \lambda\}} |\nabla v|^p dz \stackrel{(19)}{\leq} c \lambda^{-p} \sqrt{\epsilon} r^n. \quad (21)$$

Letting $\|\cdot\|$ denote the sup norm on the space of matrices,

$$\|\nabla u\| - \|\nabla v\| \leq c |S(\nabla u) - S(\nabla v)|$$

we have $\|\nabla u\| - \|\nabla v\|_{L^p(B_r(x))} \stackrel{(14)}{\leq} c \epsilon^{\frac{1}{p}} r^{\frac{n}{p}}$. So

$$\|\nabla u\|_{L^p(M_\lambda)} \leq c \|\nabla u\| - \|\nabla v\|_{L^p(B_r(x))} + c \|\nabla v\|_{L^p(M_\lambda)} \stackrel{(20)}{\leq} c \epsilon^{\frac{1}{2p}} r^{\frac{n}{p}}. \quad (22)$$

By Proposition A1 [Fr-Ja-Mu 02] there exists $c\lambda$ -Lipschitz function s such that

$$\int_{B_{\frac{r}{4}}(x)} |\nabla v - \nabla s|^p dz \leq c \int_{\{z \in B_{\frac{r}{2}}(x) : |\nabla v(z)| > \lambda\}} |\nabla v|^p dz \quad (23)$$

And so by (19)

$$\int_{B_{\frac{r}{4}}(x)} |\nabla v - \nabla s|^p dz \leq c \sqrt{\epsilon} r^n. \quad (24)$$

Let

$$\mathcal{U} := \left\{ z \in B_{\frac{r}{4}}(x) : v(z) \neq s(z) \right\}, \quad (25)$$

by Proposition A1 we also have that

$$\mathcal{U} \subset M_\lambda. \quad (26)$$

Step 1. Let $w : B_{\mathcal{E}r}(u(x)) \rightarrow \mathbb{R}^n$ be defined by $w(z) := s(u^{-1}(z))$. There exists $R \in SO(n)$

$$\int_{B_{\mathcal{E}r}(u(x))} |\nabla w - R| dz \leq \frac{c \mathcal{E}^{-n} A_u^r}{\sqrt{\ln(2 + \epsilon^{-\frac{1}{2}})}}. \quad (27)$$

Proof of Step 1. Since u is a mapping of finite dilatation and $\nabla u \in L^n(\Omega)$ by Theorem 1.2. [Cs-He-Ma 10] we have that $u^{-1} \in W^{1,1}(u(\Omega))$ and u^{-1} is a mapping of finite dilatation. Now by the BV chain rule of Ambrosio, DalMaso [Am-Da 90], (see Theorem 3.101 [Am-Fu-Pa 00] or Corollary 3.2 [Am-Da 90]) for a.e. $x \in u(\Omega)$ the restriction of s to the affine space $A(x) :=$

$u^{-1}(x) + \{\nabla u^{-1}(x)v : v \in \mathbb{R}^n\}$ is differentiable at $u^{-1}(x)$. Since for a.e. $x \in u(\Omega)$, $\det(\nabla u^{-1}(x)) > 0$ so $A(x) = \mathbb{R}^n$. Thus by Corollary 3.2 [Am-Da 90], $\nabla w(x) = \nabla s(u^{-1}(x)) \nabla u^{-1}(x)$.

Define \mathcal{J} to be the $n \times n$ diagonal matrix defined by $\mathcal{J} = \text{diag}(1, 1, \dots, 1, -1)$. Let $\mathcal{I} \in \{Id, \mathcal{J}\}$, note that for any $\mathcal{S} \subset B_{\frac{r}{4}}(x) \setminus \mathcal{U}$

$$\begin{aligned} & \int_{u(\mathcal{S})} d(\nabla w(z), SO(n)\mathcal{I}) dz \\ &= \int_{u(\mathcal{S})} d\left(\nabla s(u^{-1}(z)) \left(\nabla u(u^{-1}(z))\right)^{-1}, SO(n)\mathcal{I}\right) dz \\ &\stackrel{(25)}{=} \int_{\mathcal{S}} d\left(\nabla v(y) (\nabla u(y))^{-1}, SO(n)\mathcal{I}\right) \det(\nabla u(y)) dy. \end{aligned} \quad (28)$$

Now for any $y \in B_{\frac{r}{2}}(x)$, let $R_v(y) \in O(n)$, $R_u(y) \in SO(n)$ such that $\nabla v(y) = R_v(y)S(\nabla v(y))$, $\nabla u(y) = R_u(y)S(\nabla u(y))$. Now

$$\nabla v(y)(\nabla u(y))^{-1} = R_v(y)S(\nabla v(y))(S(\nabla u(y)))^{-1}R_u(y)^{-1}. \quad (29)$$

So

$$\begin{aligned} \left| \nabla v(y)(\nabla u(y))^{-1} - R_v(y)R_u(y)^{-1} \right| &\leq c \left| S(\nabla v(y))(S(\nabla u(y)))^{-1} - Id \right| \\ &= c \left| (S(\nabla v(y)) - S(\nabla u(y)))(S(\nabla u(y)))^{-1} \right| \\ &= c \left| (S(\nabla v(y)) - S(\nabla u(y)))(\nabla u(y))^{-1} \right|. \end{aligned} \quad (30)$$

Thus

$$\begin{aligned} & \int_{B_{\frac{r}{2}}(x)} \left| \nabla v(y)(\nabla u(y))^{-1} - R_v(y)R_u(y)^{-1} \right| \det(\nabla u(y)) dy \\ &\stackrel{(30)}{\leq} c \int_{B_{\frac{r}{2}}(x)} |S(\nabla v(y)) - S(\nabla u(y))| |\text{ADJ}(\nabla u(y))| dy \\ &\leq c \int_{B_{\frac{r}{2}}(x)} |S(\nabla v(y)) - S(\nabla u(y))| |\nabla u(y)|^{n-1} dy \\ &\leq c \|S(\nabla u) - S(\nabla v)\|_{L^p(B_{\frac{r}{2}}(x))} \left(\int_{B_{\frac{r}{2}}(x)} |\nabla u|^q dy \right)^{\frac{p-1}{p}} \\ &\stackrel{(14),(17)}{\leq} c \epsilon^{\frac{1}{p}} A_u^r r^n. \end{aligned} \quad (31)$$

Let

$$\mathcal{D} := \left\{ z \in B_{\frac{r}{2}}(x) : \det(\nabla v(z)) \leq 0 \right\}. \quad (32)$$

By (15)

$$|\mathcal{D}| \leq \epsilon r^n. \quad (33)$$

Now by (30) and the definition of \mathcal{D} we know

$$d(\nabla v(y)(\nabla u(y))^{-1}, SO(n)) \leq c \left| (S(\nabla v(y)) - S(\nabla u(y)))(\nabla u(y))^{-1} \right| \text{ for } y \in B_{\frac{r}{2}}(x) \setminus \mathcal{D}.$$

Thus taking $\mathcal{S} = B_{\frac{r}{4}}(x) \setminus (\mathcal{U} \cup \mathcal{D})$ in (28) for the case $\mathcal{I} = Id$ and applying (31) we have

$$\int_{u(B_{\frac{r}{4}}(x) \setminus (\mathcal{U} \cup \mathcal{D}))} d(\nabla w(z), SO(n)) dz \leq c \epsilon^{\frac{1}{2n}} A_u^r r^n. \quad (34)$$

Now by (30)

$$d(\nabla v(y)(\nabla u(y))^{-1}, SO(n)\mathcal{J}) \leq c \left| (S(\nabla v(y)) - S(\nabla u(y)))(\nabla u(y))^{-1} \right| \text{ for } y \in \mathcal{D} \setminus \mathcal{U}.$$

So taking $\mathcal{I} = \mathcal{J}$, $S = \mathcal{D} \setminus \mathcal{U}$ in (28) and applying (31) we have

$$\int_{u(\mathcal{D} \setminus \mathcal{U})} d(\nabla w(z), SO(n)\mathcal{J}) dz \leq c \epsilon^{\frac{1}{p}} (A_u^r)^{\frac{1}{q}} r^n.$$

Thus

$$\int_{u(\mathcal{D} \setminus \mathcal{U})} d(\nabla w(z), SO(n)) dz \leq c |u(\mathcal{D} \setminus \mathcal{U})| + c \epsilon^{\frac{1}{p}} (A_u^r)^{\frac{1}{q}} r^n. \quad (35)$$

Let $f(p) = \frac{p(n-1)}{p-1}$, so $f'(p) = -\frac{n-1}{(p-1)^2} < 0$ for all $p \in (1, n]$. So f is decreasing and $f(n) = n$ so

$$\frac{p(n-1)}{p-1} > n \text{ for all } p \in (1, n). \quad (36)$$

Now by Theorem 1.1. [Mu 90] $\int_{B_{\frac{r}{2}}(x)} \det(\nabla u(z)) \ln(2 + \det(\nabla u(z))) dz \leq C_2 A_u^r r^n$ for some constant $C_2 = C_2(A, n)$. Let $Y := \{z \in B_r(x) : \det(\nabla u(z)) > \epsilon^{-\frac{1}{2}}\}$. So $|u(\mathcal{D} \setminus Y)| \stackrel{(33)}{\leq} c \epsilon^{\frac{1}{2}} r^n$. Now $|u(Y)| = \int_Y \det(\nabla u) dz \leq c A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}})$. So $|u(\mathcal{D})| \leq C_2 A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}})$ putting this together with (35) we have $\int_{u(\mathcal{D} \setminus \mathcal{U})} d(\nabla w(z), SO(n)) dz \leq c A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}})$. So applying this to (34) we have

$$\int_{u(B_{\frac{r}{4}}(x) \setminus \mathcal{U})} d(\nabla w(z), SO(n)) dz \leq c A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}}). \quad (37)$$

Now as

$$|\mathcal{U}| \stackrel{(26)}{\leq} |M_\lambda| \stackrel{(21)}{\leq} c \lambda^{-p} \sqrt{\epsilon} r^n \quad (38)$$

so (recalling $w = s \circ u^{-1}$ and s is $c\lambda$ -Lipschitz)

$$\begin{aligned} \int_{u(\mathcal{U})} |\nabla w| dz &\leq c\lambda \int_{u(\mathcal{U})} \left| (\nabla u(u^{-1}(z)))^{-1} \right| dz \\ &= c\lambda \int_{u(\mathcal{U})} \left| (\nabla u(u^{-1}(z)))^{-1} \right| \det(\nabla u^{-1}(z)) \det(\nabla u(u^{-1}(z))) dz \\ &= c\lambda \int_{\mathcal{U}} \left| (\nabla u(y))^{-1} \right| \det(\nabla u(y)) dy \leq c\lambda \int_{\mathcal{U}} |\text{ADJ}(\nabla u(z))| dz \\ &\leq c\lambda \int_{\mathcal{U}} |\nabla u(z)|^{n-1} dz \leq c\lambda \left(\int_{\mathcal{U}} |\nabla u(z)|^q \right)^{\frac{p-1}{p}} |\mathcal{U}|^{\frac{1}{p}} \\ &\stackrel{(38), (17)}{\leq} c\lambda (A_u^r r^n)^{\frac{p-1}{p}} (c\lambda^{-p} \sqrt{\epsilon} r^n)^{\frac{1}{p}} \leq c A_u^r \epsilon^{\frac{1}{2p}} r^n. \end{aligned} \quad (39)$$

Now $\int_{B_r(x)} |\nabla u|^q dz \stackrel{(17)}{\leq} A_u^r r^n$ where $q = \frac{p(n-1)}{p-1}$. So by (36) and Holder's inequality we know that

$$\int_{B_r(x)} |\nabla u|^n dz \leq A_u^r r^n. \quad (40)$$

Let $\theta \in (0, 1)$ such that $\frac{1}{n} = \frac{\theta}{p} + \frac{1-\theta}{q}$. So by the L^p interpolation inequality (see Appendix B2 [Ev 10])

$$\begin{aligned}
\|\nabla u\|_{L^n(M_\lambda)} &\leq \|\nabla u\|_{L^p(M_\lambda)}^\theta \|\nabla u\|_{L^q(M_\lambda)}^{1-\theta} \\
&\stackrel{(17),(22)}{\leq} \left(\epsilon^{\frac{1}{2p}} r^{\frac{n}{p}} \right)^\theta \left((A_u^r)^{\frac{1}{q}} r^{\frac{n}{q}} \right)^{1-\theta} \\
&\leq \epsilon^{\frac{\theta}{2p}} A_u^r r^{n\left(\frac{\theta}{p} + \frac{1-\theta}{q}\right)} \\
&\leq \epsilon^{\frac{\theta}{2}} A_u^r r.
\end{aligned} \tag{41}$$

Thus

$$\begin{aligned}
\int_{u(B_{\frac{r}{4}}(x))} d(\nabla w(z), SO(n)) dz &= \int_{u(B_{\frac{r}{4}}(x) \setminus \mathcal{U})} d(\nabla w(z), SO(n)) dz \\
&\quad + \int_{u(\mathcal{U})} (|\nabla w(z)| + c) dz \\
&\stackrel{(39),(37)}{\leq} c|u(\mathcal{U})| + 2C_2 A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}}) \\
&\stackrel{(26)}{\leq} c \int_{M_\lambda} \det(\nabla u) dz + 3C_2 A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}}) \\
&\stackrel{(22)}{\leq} 3C_2 A_u^r r^n / \ln(2 + \epsilon^{-\frac{1}{2}}).
\end{aligned} \tag{42}$$

So in particular using (13) we have $\int_{B_{\mathcal{E}r}(u(x))} d(\nabla w(z), SO(n)) dz \leq \frac{3C_2 A_u^r \mathcal{E}^{-n}}{\ln(2 + \epsilon^{-\frac{1}{2}})}$. Thus by Proposition 2.6 [Co-Sc 06] we have that

$$\int_{B_{\mathcal{E}r}(u(x))} |\nabla w - R| dz \leq c \ln \left(\frac{\ln(2 + \epsilon^{-\frac{1}{2}})}{3C_2 A \mathcal{E}^{-n}} \right) \frac{3C_2 A_u^r \mathcal{E}^{-n}}{\ln(2 + \epsilon^{-\frac{1}{2}})} \leq \frac{c A_u^r \mathcal{E}^{-n}}{\sqrt{\ln(2 + \epsilon^{-\frac{1}{2}})}}.$$

This completes the proof of Step 1.

Step 2. We will show

$$B_{c \exp(-\frac{A_u^r}{\mathcal{E}^n})}(x) \subset u^{-1}(B_{\mathcal{E}r}(u(x))) \tag{43}$$

Proof of Step 2. Note by equation (2.5) of the proof of Theorem 1 of [Ma 94] we know that for any $y \in B_{\frac{r}{2}}(x)$, $h \in (0, \frac{r}{2}]$

$$\begin{aligned}
\text{osc}_{B_h(y)} u &\leq c \left(\log \left(\frac{r}{2h} \right) \right)^{-\frac{1}{n}} \left(\int_{B_{\frac{r}{2}}(y)} |\nabla u|^n dz \right)^{\frac{1}{n}} \\
&\stackrel{(40)}{\leq} c (A_u^r)^{\frac{1}{n}} r \left(\log \left(\frac{r}{2h} \right) \right)^{-\frac{1}{n}}.
\end{aligned} \tag{44}$$

We claim

$$\text{dist} \left(x, u^{-1}(\partial B_{\mathcal{E}r}(u(x))) \right) \geq c \exp \left(-\frac{A_u^r}{\mathcal{E}^n} \right). \tag{45}$$

So see this pick $z \in u^{-1}(\partial B_{\mathcal{E}r}(u(x))) \cap B_{\frac{r}{2}}(x)$, since u is a homeomorphism $\mathcal{E}r = |u(z) - u(x)| \leq c(A_u^r)^{\frac{1}{n}} r \left(\log \left(\frac{r}{2|z-x|} \right) \right)^{-\frac{1}{n}}$. Thus

$$\left(\log \left(\frac{r}{2|z-x|} \right) \right)^{\frac{1}{n}} \mathcal{E} \leq c(A_u^r)^{\frac{1}{n}}$$

and so $\log \left(\frac{r}{2|z-x|} \right) \mathcal{E}^n \leq cA_u^r$ and hence $\frac{r}{2|z-x|} \leq c \exp(\frac{A_u^r}{\mathcal{E}^n})$ and finally $cr \exp(-\frac{A_u^r}{\mathcal{E}^n}) \leq |z-x|$ which establishes (43).

Proof of Lemma completed. Note that if matrix B satisfies $\|B\|^n \leq Q \det(B)$ then as $\Lambda(B) := \inf \{|Bv| : v \in S^{n-1}\}$. If we let B_I be the smallest number such that $\det(B) \leq B_I \Lambda(B)^n$ it is well known (see for example [Va 71] p44) $B_I \leq Q^{n-1}$ so

$$\Lambda(B) \geq \frac{\det(B)^{\frac{1}{n}}}{Q^{\frac{n-1}{n}}}. \quad (46)$$

So it is an exercise to see

$$|AB| \geq \frac{(\det(B))^{\frac{1}{n}} |A|}{Q^{\frac{n-1}{n}} n} \geq \frac{Q^{-1}}{n^2} |B| |A| \text{ for any } A \in \mathbb{R}^{n \times n}. \quad (47)$$

Recall u is of integrable dilatation and so we have function K such that $\|\nabla u(z)\|^n \leq K(z) \det(\nabla u(z))$. Let

$$\mathbb{P} := \left\{ z \in B_r(x) : |K(z)| \geq \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{\frac{1}{8(n-1)}} \right\}. \quad (48)$$

Now as $\nabla w(z) = \nabla s(u^{-1}(z))(\nabla u(u^{-1}(z)))^{-1}$. Thus by (27) Step 1

$$\begin{aligned} \frac{cA_u^r r^n}{\sqrt{\ln(2 + \epsilon^{-\frac{1}{2}})}} &\stackrel{(27)}{\geq} \int_{B_{\mathcal{E}r}(x)} |\nabla w - R| \det(\nabla u^{-1}(z)) \det(\nabla u(u^{-1}(z))) dz \\ &= \int_{u^{-1}(B_{\mathcal{E}r}(x))} |\nabla s(z)(\nabla u(z))^{-1} - R| \det(\nabla u(z)) dz \\ &\geq \int_{u^{-1}(B_{\mathcal{E}r}(x)) \setminus \mathbb{P}} |(\nabla s(z) - R \nabla u(z)) (\nabla u(z))^{-1}| \det(\nabla u(z)) dz \\ &\geq c \int_{u^{-1}(B_{\mathcal{E}r}(x)) \setminus \mathbb{P}} |(\nabla s(z) - R \nabla u(z)) \text{ADJ}(\nabla u(z))| dz. \end{aligned} \quad (49)$$

So by using (46) $\|\nabla u(z)^{-1}\|^n \leq K(z)^{n-1} \det((\nabla u(z))^{-1})$, so

$$\|\text{ADJ}(\nabla u(z))\|^n \leq K(z)^{n-1} \det(\text{ADJ}(\nabla u(z))).$$

Hence if $z \notin \mathbb{P}$ by (47),

$$|(\nabla s(z) - R \nabla u(z)) \text{ADJ}(\nabla u(z))| \geq n^{-2} \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{8}} |(\nabla s(z) - R \nabla u(z))| |\text{ADJ}(\nabla u(z))|. \quad (50)$$

Thus by (49), (50)

$$\frac{cA_u^r r^n}{\left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{\frac{3}{8}}} \geq \int_{u^{-1}(B_{\mathcal{E}r}(x)) \setminus \mathbb{P}} |\nabla s(z) - R \nabla u(z)| |\text{ADJ}(\nabla u(z))| dz. \quad (51)$$

Now let $\mathcal{F} := \left\{ z \in B_r(x) : |\text{ADJ}(\nabla u(z))| < \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{4}} \right\}$ so

$$\int_{u^{-1}(B_{\mathcal{E}r}(x)) \setminus (\mathcal{F} \cup \mathbb{P})} |\nabla s(z) - R\nabla u(z)| dz \leq \frac{cA_u^r r^n}{\left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{\frac{1}{8}}}. \quad (52)$$

For any matrix $A \in \mathbb{R}^{n \times n}$ let $M_{ij}(A)$ is the i, j minor of A . Thus $|M_{ij}(\nabla u(z))| < \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{4}}$ for any $z \in \mathcal{F}$. So $|\det(\nabla u(z))| \leq c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{4}} |\nabla u(z)|$. Thus

$$\begin{aligned} \int_{\mathcal{F}} |\det(\nabla u(z))| dz &\leq c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{4}} \int_{\mathcal{F}} |\nabla u(z)| dz \\ &\stackrel{(40)}{\leq} c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{4}} (A_u^r)^{\frac{1}{n}} r^n. \end{aligned} \quad (53)$$

Now as $\|\nabla u(z)\|^n \leq K(z) \det(\nabla u(z))$ for a.e. $z \in B_r(x)$. Note by (48) $|\mathbb{P}| \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{\frac{1}{8(n-1)}} \leq \int K dz \leq c$ and thus

$$|\mathbb{P}| < c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{8(n-1)}}. \quad (54)$$

So if $z \notin \mathbb{P}$ then $\|\nabla u(z)\|^n \leq c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{\frac{1}{8(n-1)}} \det(\nabla u(z))$. Thus

$$\begin{aligned} \int_{\mathcal{F} \setminus \mathbb{P}} |\nabla u|^n dz &\leq c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{\frac{1}{8(n-1)}} \int_{\mathcal{F} \setminus \mathbb{P}} \det(\nabla u(z)) dz \\ &\stackrel{(53)}{\leq} c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{8}} (A_u^r)^{\frac{1}{n}} r^n. \end{aligned} \quad (55)$$

Now

$$\int_{\mathbb{P}} |\nabla u| dz \leq c |\mathbb{P}|^{\frac{n-1}{n}} \left(\int_{B_r(x)} |\nabla u|^n dz \right)^{\frac{1}{n}} \stackrel{(40), (54)}{\leq} c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{8n}} (A_u^r)^{\frac{1}{n}} r. \quad (56)$$

So

$$\int_{\mathcal{F} \cup \mathbb{P}} |\nabla u| dz \leq \int_{\mathbb{P}} |\nabla u| dz + \int_{\mathcal{F} \setminus \mathbb{P}} |\nabla u| dz \stackrel{(56), (55)}{\leq} c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{16n}} (A_u^r)^{\frac{1}{n^2}} r. \quad (57)$$

By using Holder's inequality we see

$$\begin{aligned} \|\nabla s\|_{L^1(\mathcal{F} \cup \mathbb{P})} &\leq c \|S(\nabla u) - S(\nabla v)\|_{L^p(B_{\frac{r}{2}}(x))} + c \|\nabla u\|_{L^1(\mathcal{F} \cup \mathbb{P})} + c \|\nabla v - \nabla s\|_{L^p(B_{\frac{r}{4}}(x))} \\ &\stackrel{(57), (24), (14)}{\leq} c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{16n}} (A_u^r)^{\frac{1}{n^2}} r. \end{aligned} \quad (58)$$

Thus $\int_{\mathcal{F} \cup \mathbb{P}} |\nabla s - R\nabla u| dz \stackrel{(57), (58)}{\leq} c \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{16n}} (A_u^r)^{\frac{1}{n^2}} r$. Hence putting this together with (52), (24) we have

$$\int_{u^{-1}(B_{\mathcal{E}r}(x))} |\nabla v - R\nabla u| dz \leq cA_u^r \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{16n}} r. \quad (59)$$

And putting this together with (43) we have

$$\int_{B_{c \exp(-\frac{A_u^r}{\mathcal{E}^n})}(x)} |\nabla v - R\nabla u| dz \leq cr^{-n+1} \exp\left(\frac{nA_u^r}{\mathcal{E}^n}\right) A_u^r \left(\ln(2 + \epsilon^{-\frac{1}{2}}) \right)^{-\frac{1}{16n}}. \quad (60)$$

Note that since $r^{-32n(n-1)} \leq \ln\left(2 + \epsilon^{-\frac{1}{2}}\right)$ so $\left(\ln\left(2 + \epsilon^{-\frac{1}{2}}\right)\right)^{-\frac{1}{32n}} \leq r^{n-1}$ so putting this together with (60) we have established (16).

2.1. Proof of Theorem 1 completed. By Theorem 1 [Ma-Vi 98] u is a discrete open mapping. Let B_u denote the set of points $z \in B_r(x)$ such that u is not locally invertible in any neighborhood containing z . By definition this is a closed set.

Step 1. We will show $u(B_r(x) \setminus B_u)$ is connected.

Proof of Step 1. By a theorem of Chernavskii B_u [Ch 64], (also see [Va 66]) B_u has topological dimension at most $n - 2$. By Example VI 11 p93 [Wa 41] we know $B_r(x)$ can not be separated by B_u and so $B_r(x) \setminus B_u$ is connected.

Proof of Theorem 1. For any $z \in B_r(x) \setminus B_u$ by definition of B_u there exists $s_z > 0$ such that $u|_{B_{s_z}(z)}$ is injective, therefor by applying Lemma 1 we know that for some $r_z \in (0, s_z)$ there exists $R_z \in SO(n)$ such that $\nabla u = R_z \nabla v$ on $B_{r_z}(z)$.

Pick $z_0 \in B_r(x) \setminus B_u$. Let $z_1 \in B_r(x) \setminus B_u$, $z_1 \neq z_0$. Since $B_r(x) \setminus B_u$ is connected and open and is therefor arcwise connected so there exists a homomorphism $\psi : [0, 1] \rightarrow B_r(x) \setminus B_u$ with $\psi(0) = z_0$, $\psi(1) = z_1$. For each $z \in B_r(x) \setminus B_u$ let

$$\alpha_z := \sup \left\{ \alpha > 0 : u|_{B_\alpha(z)} \text{ is injective} \right\}.$$

It is clear $\beta = \inf \{ \alpha_z : z \in \psi([0, 1]) \} > 0$ since otherwise by compactness $B_u \cap \psi([0, 1]) \neq \emptyset$. Let

$$\mathcal{G} := \left\{ h \in [0, 1] : \nabla u(z) = R_{z_0} \nabla v(z) \text{ for a.e. } z \in \bigcup_{\gamma \in [0, h]} B_{\frac{\beta}{2}}(\psi(\gamma)) \right\}. \quad (61)$$

It is clear \mathcal{G} is a closed set, it is also straightforward to see it is open because if $h \in \mathcal{G}$ there exists $\tilde{R} \in SO(n)$ such that $\nabla u(x) = \tilde{R} \nabla v(x)$ for a.e. $x \in B_{\frac{\beta}{2}}(\psi(h))$. Since we also know $\nabla u(z) = R_{z_0} \nabla v(z)$ for a.e. $z \in B_{\frac{\beta}{2}}(\psi(h))$ it is clear $\tilde{R} = R_{z_0}$ and thus there exists $\delta > 0$ with $(h - \delta, h + \delta) \subset \mathcal{G}$. As \mathcal{G} is open and closed in $[0, 1]$ and as it is non empty we have that $\mathcal{G} = [0, 1]$. In particular this implies that for every $z \in B_r(x) \setminus B_u$, $\nabla u(y) = R_{z_0} \nabla v(y)$ for a.e. $y \in B_{\frac{\beta}{2}}(z)$. Thus $\nabla u(z) = R_{z_0} \nabla v(z)$ for a.e. $z \in B_r(x) \setminus B_u$. Since B_u has dimension at most $n - 2$ we know $|B_u| = 0$ therefor (3) follows immediately. \square

3. PRELIMINARY LEMMAS FOR THEOREM 3

Lemma 2. Let $r \in (0, 1)$, $A > 1$, $p \in [1, n]$, $q = \frac{p(n-1)}{p-1}$, $\mathcal{E} \in (0, 1)$. Suppose $v \in W^{1,p}(B_r(x) : \mathbb{R}^n)$ and $u \in W^{1,q}(B_r(x) : \mathbb{R}^n)$ is a homeomorphism of integrable dilatation. There exists small constant $\epsilon_0 = \epsilon_0(A, r, \mathcal{E})$ such that if functions u, v satisfy

$$\int_{B_r(x)} |S(\nabla u) - S(\nabla v)|^p dz \leq \epsilon, \quad (62)$$

$$\int_{B_r(x)} |\text{sgn}(\det(\nabla v(z))) - 1| dz \leq \epsilon \quad (63)$$

$$\int_{B_r(x)} |\nabla u|^q dz \leq A \quad (64)$$

for $\epsilon < \epsilon_0$ and there exists $\Xi \subset B_{\frac{r}{2}}(x)$ such that

$$|B_{\frac{r}{2}}(x) \setminus \Xi| \leq \frac{C_0^n}{16^n} r^n \exp\left(-\frac{nA}{\mathcal{E}^n}\right) \quad (65)$$

and

$$B_{\mathcal{E}h}(u(x)) \subset u(B_{\frac{h}{4}}(x)) \text{ for any } x \in \Xi, h \in \left[\frac{C_0}{8} r \exp\left(-\frac{A}{\mathcal{E}^n}\right), \frac{r}{2} \right]. \quad (66)$$

Then there exists $C_2 = C_2(n, \int_{B_r(x)} K dz)$ and $R \in SO(n)$ such that

$$\int_{B_{\frac{r}{2}}(x)} |\nabla u - R \nabla v| dz \leq C_2 \mathcal{E}^{-n} A^{3n} \exp\left(\frac{2^{n+2} n^3 A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}\right)\right)^{-\frac{1}{64n^2}} r^n.$$

Proof of Lemma 2. To simplify notation let $\Lambda_{\mathcal{E}}^A = \exp\left(-\frac{A}{\mathcal{E}^n}\right)$. Note by (65)

$$B_{\frac{r}{2}}(x) \subset \bigcup_{x \in \Xi \cap B_{\frac{r}{2}}(x)} B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x). \quad (67)$$

So by Theorem 2.7 [Ma 95] we can extract some finite collection

$$\left\{ B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_1), B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_2), \dots, B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_P) \right\}$$

where

$$B_{\frac{r}{2}}(x) \subset \bigcup_{i=1}^P B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_i) \quad (68)$$

and

$$\sum_{i=1}^P \mathbb{1}_{B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_i)} \leq c. \quad (69)$$

Now for any $i, j \in \{1, 2, \dots, P\}$ if we have $B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_i) \cap B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_j) \neq \emptyset$ then

$$B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_i) \subset B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(x_j). \quad (70)$$

We assume we order the balls such that $B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_{i+1}) \cap B_{\frac{C_0}{8} r \Lambda_{\mathcal{E}}^A}(x_i) \neq \emptyset$ for each $i = 1, 2, \dots, P-1$.

Since $\int_{B_{\frac{r}{2}}(x_i)} |S(\nabla u) - S(\nabla v)|^p dz \leq 2^n \epsilon$, $\int_{B_{\frac{r}{2}}(x_i)} |\operatorname{sgn}(\det(\nabla v)) - 1| dz \leq 2^n \epsilon$, $\int_{B_{\frac{r}{2}}(x_i)} |\nabla u|^q dz \leq 2^n A$. By applying Lemma 1 on each ball $B_{\frac{r}{2}}(x_i)$ we have that for each $i \in \{1, 2, \dots, P\}$ there exists $R_i \in SO(n)$ such that

$$\int_{B_{\frac{C_0}{2} r \Lambda_{\mathcal{E}}^A}(x_i)} |\nabla v - R_i \nabla u| dz \leq C_1 A \exp\left(\frac{2^n n A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}\right)\right)^{-\frac{1}{32n}}. \quad (71)$$

Now by (70) $\left| B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(x_{i+1}) \cap B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(x_i) \right| \geq \frac{C_0^n}{8^n} r^n (\Lambda_{\mathcal{E}}^A)^n$ and so by (65) there must exists

$$\omega_0 \in B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(x_{i+1}) \cap B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(x_i) \cap \Xi.$$

So as

$$B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(\omega_0) \subset B_{\frac{C_0}{2} r \Lambda_{\mathcal{E}}^A}(x_i) \cap B_{\frac{C_0}{2} r \Lambda_{\mathcal{E}}^A}(x_{i+1}) \quad (72)$$

by definition of Ξ we have that

$$B_{\frac{\mathcal{E} C_0}{8} r \Lambda_{\mathcal{E}}^A}(u(\omega_0)) \subset u\left(B_{\frac{C_0}{32} r \Lambda_{\mathcal{E}}^A}(\omega_0)\right). \quad (73)$$

Let C_{iso} denote the constant of the isoperimetric inequality in \mathbb{R}^n . We claim (73) implies

$$\int_{B_{\frac{C_0}{4} r \Lambda_{\mathcal{E}}^A}(\omega_0)} |\nabla u|^{n-1} dz \geq \frac{\mathcal{E}^{n-1} C_0^n}{n^3 C_{iso} 128^{n-1}} r^n (\Lambda_{\mathcal{E}}^A)^n. \quad (74)$$

Suppose this is not true. Define $\psi : u \left(B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(\omega_0) \right) \rightarrow \mathbb{R}$ by $\psi(z) = |u^{-1}|$. Since by Theorem 4.1 [He-Ko-Ma 06] we know $u^{-1} \in W^{1,n}$ we know that $\psi \in W^{1,n}$. So either by considering difference quotients or by applying the Chain rule of [Am-Da 90] we have

$$|\nabla \psi(z)| \leq \|\nabla u(u^{-1}(z))^{-1}\| = \|\text{ADJ}(\nabla u(u^{-1}(z)))\| \det(\nabla u^{-1}(z)). \quad (75)$$

So by the Co-area formula we have

$$\begin{aligned} \int_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}^{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A} H^{n-1}(\psi^{-1}(t)) dt &= \int_u \left(B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(\omega_0) \right) |\nabla \psi(z)| dz \\ &\stackrel{(75)}{\leq} n \int_u \left(B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(\omega_0) \right) |\text{ADJ}(\nabla u(u^{-1}(z)))| \det(\nabla u^{-1}(z)) dz \\ &= n \int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(\omega_0)} |\text{ADJ}(\nabla u(y))| dy \\ &\leq n^3 \int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0)} |\nabla u(y)|^{n-1} dy. \end{aligned}$$

So since we are assuming (74) is false there must exists $t \in \left(\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A, \frac{C_0}{4}r\Lambda_{\mathcal{E}}^A \right)$ such that

$$H^{n-1}(\psi^{-1}(t)) \leq \frac{8(\mathcal{E}C_0)^{n-1}}{128^{n-1}C_{iso}} (r\Lambda_{\mathcal{E}}^A)^{n-1}. \quad (76)$$

However by construction $\psi^{-1}(t) = u(\partial B_t(\omega_0))$. Now note that by the isoperimetric inequality we have

$$\begin{aligned} |u(B_t(\omega_0))|^{\frac{n-1}{n}} &\leq C_{iso} H^{n-1}(\partial u(B_t(\omega_0))) \\ &\stackrel{(76)}{\leq} \frac{8(\mathcal{E}C_0)^{n-1}}{128^{n-1}} (r\Lambda_{\mathcal{E}}^A)^{n-1}. \end{aligned} \quad (77)$$

Hence

$$\begin{aligned} \mathcal{E}^n \frac{C_0^n}{8^n} r^n (\Lambda_{\mathcal{E}}^A)^n &\stackrel{(73)}{\leq} \left| u \left(B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(\omega_0) \right) \right| \\ &\stackrel{(77)}{\leq} 8^{\frac{n}{n-1}} \frac{(\mathcal{E}C_0)^n}{128^n} (r\Lambda_{\mathcal{E}}^A)^n \end{aligned} \quad (78)$$

which is a contradiction. Thus (74) is established.

So by (71) and (72) we have

$$\begin{aligned} C_1 A \exp \left(\frac{2^n n A}{\mathcal{E}^n} \right) \left(\ln \left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n} \right) \right)^{-\frac{1}{32n}} (\Lambda_{\mathcal{E}}^A)^n r^n C_0^n \\ \geq \int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0)} |(R_i - R_{i+1}) \nabla u| dz. \end{aligned} \quad (79)$$

Let

$$\mathcal{O} = \left\{ z \in B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) : K(z) < \left(\ln \left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n} \right) \right)^{\frac{1}{64n}} \right\}.$$

So

$$\left| B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus \mathcal{O} \right| \leq c \left(\ln(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}) \right)^{-\frac{1}{64n}}. \quad (80)$$

Hence

$$\begin{aligned} \int_{\mathcal{O}} |\nabla u|^{n-1} dz &\geq \int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0)} |\nabla u|^{n-1} dz - \int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus \mathcal{O}} |\nabla u|^{n-1} dz \\ &\stackrel{(74)}{\geq} \frac{\mathcal{E}^{n-1} C_0^n}{n^3 C_{iso} 128^{n-1}} r^n (\Lambda_{\mathcal{E}}^A)^n - \left(\int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0)} |\nabla u|^q dz \right)^{\frac{n-1}{q}} \left| B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0) \setminus \mathcal{O} \right|^{\frac{q-1}{q}} \\ &\stackrel{(80)}{\geq} \frac{\mathcal{E}^{n-1} C_0^n}{n^3 C_{iso} 256^{n-1}} r^n (\Lambda_{\mathcal{E}}^A)^n \end{aligned} \quad (81)$$

And by (47)

$$|(R_i - R_{i+1}) \nabla u(z)| \geq n^{-2} \left(\ln(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}) \right)^{-\frac{1}{64n}} |(R_i - R_{i+1})| |\nabla u(z)| \text{ for any } z \in \mathcal{O}.$$

Putting this together with (79) we have

$$\begin{aligned} n^2 C_1 A \exp \left(\frac{2^n n A}{\mathcal{E}^n} \right) \left(\ln(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}) \right)^{-\frac{1}{64n}} (\Lambda_{\mathcal{E}}^A)^n r^n C_0^n \\ \geq \int_{\mathcal{O}} |R_i - R_{i+1}| |\nabla u| dz. \end{aligned} \quad (82)$$

Now by (64) $\int_{B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0)} |R_i - R_{i+1}|^q |\nabla u|^q dz \leq n^{2q} A r^n$. Let $\theta = \frac{q(n-2)}{(n-1)(q-1)}$, so $\frac{1}{n-1} = \frac{\theta}{q} + 1 - \theta$. Note $1 - \theta = \frac{q-n+1}{(n-1)(q-1)}$. Now letting $\tau(q) = \frac{q-n+1}{(n-1)(q-1)}$ we have $\tau'(q) = \frac{n-2}{(n-1)(q-1)^2} \geq 0$ for all $q \geq n$. So $(1 - \theta) \geq \frac{1}{(n-1)^2}$ for all $q \geq n$. By the L^p interpolation inequality (see Appendix B2 [Ev 10]) since $r^{\frac{\theta}{q}} r^{n(1-\theta)} = r^{\frac{n}{n-1}}$ we know

$$\begin{aligned} &\|(R_i - R_{i+1}) |\nabla u|\|_{L^{n-1}(\mathcal{O})} \\ &\leq \|(R_i - R_{i+1}) |\nabla u|\|_{L^q(B_{\frac{C_0}{4}r\Lambda_{\mathcal{E}}^A}(\omega_0))}^{\theta} \|(R_i - R_{i+1}) |\nabla u|\|_{L^1(\mathcal{O})}^{1-\theta} \\ &\leq c A r^{\frac{\theta}{q}} \|(R_i - R_{i+1}) |\nabla u|\|_{L^1(\mathcal{O})}^{1-\theta} \\ &\stackrel{(82)}{\leq} c A^2 r^{\frac{n}{n-1}} \exp \left(\frac{2^n n^2 A}{\mathcal{E}^n} \right) \left(\ln(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}) \right)^{-\frac{1}{64(n-1)^2 n}}. \end{aligned} \quad (83)$$

So

$$\begin{aligned} |R_i - R_{i+1}| \frac{\mathcal{E}^{n-1} C_0^n r^n (\Lambda_{\mathcal{E}}^A)^n}{256^{n-1} n^3 C_{iso}} &\stackrel{(81)}{\leq} |R_i - R_{i+1}| \int_{\mathcal{O}} |\nabla u|^{n-1} dz \\ &\stackrel{(83)}{\leq} c r^n A^{2n} \exp \left(\frac{2^n n^3 A}{\mathcal{E}^n} \right) \left(\ln(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}) \right)^{-\frac{1}{64n^2}}. \end{aligned}$$

Hence

$$|R_i - R_{i+1}| \leq c\mathcal{E}^{-n} A^{2n} \exp\left(\frac{2^{n+1}n^3 A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}\right)\right)^{-\frac{1}{64n^2}}.$$

Now since from (69)

$$P \leq c \left(\Lambda_{\mathcal{E}}^A\right)^{-n}. \quad (84)$$

we know

$$|R_1 - R_i| \leq c\mathcal{E}^{-n} A^{2n} \exp\left(\frac{2^{n+2}n^3 A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}\right)\right)^{-\frac{1}{64n^2}}. \quad (85)$$

So

$$\begin{aligned} \int_{B_{\frac{r}{2}}(x)} |\nabla v - R_1 \nabla u| dz &\stackrel{(68)}{\leq} \sum_{i=1}^P \int_{B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(x_i)} |\nabla v - R_1 \nabla u| dz \\ &\leq \sum_{i=1}^P \int_{B_{\frac{C_0}{8}r\Lambda_{\mathcal{E}}^A}(x_i)} |\nabla v - R_i \nabla u| dz + |(R_1 - R_i) \nabla u| dz \\ &\stackrel{(71),(84),(85)}{\leq} cA \exp\left(\frac{2^n n A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}\right)\right)^{-\frac{1}{32n}} r^n \\ &\quad + c\mathcal{E}^{-n} A^{2n} \exp\left(\frac{2^{2n+2}n^3 A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^{n+2}}\right)\right)^{-\frac{1}{64n^2}} \int_{B_r(x)} |\nabla u| dz \\ &\stackrel{(64)}{\leq} c\mathcal{E}^{-n} A^{3n} \exp\left(\frac{2^{n+2}n^3 A}{\mathcal{E}^n}\right) \left(\ln\left(2 + \frac{\epsilon^{-\frac{1}{2}}}{2^n}\right)\right)^{-\frac{1}{64n^2}} r^n. \end{aligned} \quad (86)$$

Lemma 3. Suppose we have measurable $K : \Omega \rightarrow \mathbb{R}_+$ and $u_k \in W^{1,n}(\Omega : \mathbb{R}^n)$ is an equibounded sequence with

$$\|\nabla u_k\|^n \leq K \det(\nabla u_k) \text{ for all } k \quad (87)$$

and (u_k) converges weakly in $W^{1,n}$ to u . For a.e. $x \in \Omega$ there exists $r_x > 0$ and $N_x \in \mathbb{N}$ such that $u|_{B_{r_x}(x)}$ and $u_k|_{B_{r_x}(x)}$ are injective for all $k \geq N_x$

$$B_{r_x \det(\nabla u(x))^{\frac{1}{n}} / 2K(x)^{\frac{n-1}{n}}}(u(x)) \subset u(B_{r_x}(x)) \cap u_k(B_{r_x}(x)), \quad (88)$$

Proof of Lemma 3. We know that for a.e. $x \in \Omega$, $\det(\nabla u(x)) > 0$ and by Theorem 1.4. [Ge-Iw 99] we have that u is a quasiregular and satisfies $\|\nabla u(x)\|^n \leq K(x) \det(\nabla u(x))$. For any matrix $A \in \mathbb{R}^{n \times n}$ let $\Lambda(A) = \inf_{v \in S^{n-1}} |Av|$. By (46)

$$\Lambda(\nabla u(x)) \geq \frac{\det(\nabla u(x))^{\frac{1}{n}}}{K^{\frac{n-1}{n}}(x)}. \quad (89)$$

Pick x for which $\nabla u(x)$ exists and $\det(\nabla u(x)) > 0$. Let $\delta = \frac{(\det(\nabla u(x)))^{\frac{1}{n}}}{100K(x)^{\frac{n-1}{n}}}$. Let $L(z) = \nabla u(x)(z - x) + u(x)$. Let $\mathcal{S}(x) = \min \{\|\nabla u(x)\|, \|\nabla u(x)\|^{-1}\}$. We can find $\tau_x > 0$ such that

$$|u(z) - L(z)| \leq \frac{\delta^3}{2} \mathcal{S}(x) |z - x| \text{ for } z \in B_{\tau_x}(x). \quad (90)$$

As we have seen before in Lemma 1, [Ma 94] we know that for any compact subset $\tilde{\Omega} \subset\subset \Omega$, letting $d(\tilde{\Omega}, \partial\Omega) = \sigma$ we have

$$\begin{aligned} \text{osc}_{B_h(y)} u_k &\leq c \left(\log \left(\frac{\sigma}{h} \right) \right)^{-\frac{1}{n}} \left(\int_{B_{\sigma}(y)} |\nabla u_k|^n \right)^{\frac{1}{n}} \\ &\leq c \left(\log \left(\frac{\sigma}{h} \right) \right)^{-\frac{1}{n}} \text{ for any } k. \end{aligned} \quad (91)$$

Hence the sequence is equi-continuous and

$$u_k \xrightarrow{L^\infty(\tilde{\Omega})} u \text{ for any } \tilde{\Omega} \subset\subset \Omega. \quad (92)$$

So we can find $N_x \in \mathbb{N}$ such that for every $k \geq N_x$,

$$|u_k(z) - L(z)| \leq \delta^3 \mathcal{S}(x) |z - x| \text{ for } z \in B_{\tau_x}(x). \quad (93)$$

Note that since $\Lambda(\nabla u(x)) \stackrel{(89)}{\geq} 100\delta$. So $B_{100\delta\tau_x}(L(x)) \subset L(B_{\tau_x}(x))$ Let $\Psi_t(z) = tL(z) + (1-t)u_k(z)$ so by (93)

$$\Psi_t(\partial B_{\tau_x}(x)) \subset N_{\delta^3\tau_x}(L(\partial B_{\tau_x}(x))) \text{ for every } t \in [0, 1]. \quad (94)$$

And in particular

$$\Psi_t(\partial B_{\tau_x}(x)) \cap B_{99\delta\tau_x}(u(x)) = \emptyset \text{ for every } t \in [0, 1]. \quad (95)$$

Thus $\deg(u_k, B_{\tau_x}(x), z) = \deg(L, B_{\tau_x}(x), z) = 1$ for every $t \in [0, 1], z \in B_{99\delta\tau_x}(u(x))$.

Now $\|\nabla L\| = \|\nabla u(x)\|$ so $L(B_{\frac{99\delta\mathcal{S}(x)\tau_x}{2}}(x)) \subset L(B_{\frac{99\delta\tau_x\|\nabla u(x)\|}{2}}(x)) \subset B_{\frac{99\delta\tau_x}{2}}(u(x))$. So by (93) we know that for $k \geq N_x$

$$u_k \left(\partial B_{\frac{99\delta\mathcal{S}(x)\tau_x}{2}}(x) \right) \subset N_{\delta^3\mathcal{S}(x)\tau_x} \left(L \left(\partial B_{\frac{99\delta\mathcal{S}(x)\tau_x}{2}}(x) \right) \right) \subset B_{99\delta\tau_x}(u(x)),$$

thus $u_k|_{B_{\frac{99\delta\mathcal{S}(x)\tau_x}{2}}(x)}$ is injective. By the same argument $u|_{B_{\frac{99\delta\mathcal{S}(x)\tau_x}{2}}(x)}$ is injective so defining $r_x = \frac{99\delta\mathcal{S}(x)\tau_x}{2}$ establishes the first part of the lemma.

Now by definition of Λ we know $B_{\Lambda(\nabla u(x))r_x}(u(x)) \subset L(B_{r_x}(0))$ and by (89) and definition of $\delta > 0$ we have $\Lambda(\nabla u(x)) \geq 100\delta$ and so $\delta^3\mathcal{S}(x) \leq \left(\frac{\Lambda(\nabla u(x))}{100} \right)^3$, thus by (90), (93) $B_{\frac{\Lambda(\nabla u(x))r_x}{2}}(u(x)) \subset u(B_{r_x}(x)) \cap u_k(B_{r_x}(x))$ hence by (89), (88) follows.

Lemma 4. Let $p \in [1, n]$, $q = \frac{p(n-1)}{p-1}$. Suppose (v_k) is an equibounded sequence in $W^{1,p}(\Omega : \mathbb{R}^n)$ and (u_k) an equibounded sequence in $W^{1,q}(\Omega : \mathbb{R}^n)$. Let $K : \Omega \rightarrow \mathbb{R}_+$ be a measurable function and assume sequence (u_k) satisfies $\det(\nabla u_k(x)) > 0$ and $\|\nabla u_k(x)\|^n \leq K(x) \det(\nabla u_k(x))$ for a.e. $x \in \Omega$, for any $k \in \mathbb{N}$. Assume also that $\text{sgn}(\det(\nabla v_k)) \xrightarrow{L^1} 1$,

$$\int_{\Omega} |S(\nabla u_k) - S(\nabla v_k)|^p dz \rightarrow 0 \text{ as } k \rightarrow \infty \quad (96)$$

and $u_k \xrightarrow{W^{1,n}} u$, $v_k \xrightarrow{W^{1,1}} v$. Then for a.e. $x \in \Omega$, $\det(\nabla u(x)) > 0$ and there exists $R_x \in SO(n)$ such that $R_x \nabla v(x) = \nabla u(x)$. Consequently $S(\nabla u(x)) = S(\nabla v(x))$ and $\det(\nabla v(x)) > 0$ for a.e. $x \in \Omega$.

Proof of Lemma 4.

Step 1. Let $\omega > 0$. For a.e. $x \in \Omega$ there exists $w_x > 0$ such that for any $\tau \in (0, w_x)$ we can find $N_\tau \in \mathbb{N}$ with the property that if $k \geq N_\tau$ then for $R_k \in SO(n)$ we have

$$\int_{B_\tau(x)} |\nabla u_k - R_k \nabla v_k| dz \leq \omega. \quad (97)$$

Proof of Step 1. By Lemma 3 for a.e. $x \in \Omega$ there exists $r_x > 0$, $N_x \in \mathbb{N}$ such that for every $k > N_x$, $u|_{B_{r_x}(x)}$ and $u_k|_{B_{r_x}(x)}$ are injective and

$$B_{r_x \det(\nabla u(x))^{\frac{1}{n}} / 8K(x)^{\frac{n-1}{n}}}(u(x)) \subset u(B_{\frac{r_x}{4}}(x)) \cap u_k(B_{\frac{r_x}{4}}(x)). \quad (98)$$

So by (92) we can assume N_x was chosen large enough so that

$$B_{r_x \det(\nabla u(x))^{\frac{1}{n}} / 16K(x)^{\frac{n-1}{n}}}(u_k(x)) \subset u(B_{\frac{r_x}{4}}(x)) \cap u_k(B_{\frac{r_x}{4}}(x)). \quad (99)$$

Let $\mathcal{E}_x = \min \left\{ 1, \frac{\det(\nabla u(x))^{\frac{1}{n}}}{16K(x)^{\frac{n-1}{n}}} \right\}$. Now u_k is equibounded in $W^{1,n}$ so let C_1 be such that

$$\sup_k \int_{\Omega} |\nabla u_k|^n dz \leq C_1.$$

Let

$$\tau_m = \max \left\{ \int_{B_{r_x}(x)} |S(\nabla u_m) - S(\nabla v_m)|^p dz, \int_{B_{r_x}(x)} |\operatorname{sgn}(\det(\nabla v_m)) - 1| dz \right\}$$

so $\tau_m \rightarrow 0$ as $m \rightarrow \infty$. So applying Lemma 1 on $B_{r_x}(x)$ we have that for $A_x := \frac{C_1}{r_x^n}$ we have for some $R_m \in SO(n)$

$$\int_{B_{C_0 r_x \exp(-\frac{A_x}{\mathcal{E}_x})}(x)} |\nabla u_m - R_m \nabla v_m| dz \leq C_1 A_x \exp\left(\frac{n A_x}{\mathcal{E}_x}\right) \ln\left(2 + \tau_m^{-\frac{1}{4}}\right)^{-\frac{1}{32n}}.$$

So defining $w_x = C_0 r_x \exp\left(-\frac{A_x}{\mathcal{E}_x}\right)$. For any $\tau \in (0, w_x)$ we can find N_τ such that for $k \geq N_\tau$ for some $R_k \in SO(n)$ (97) holds true.

Step 2. For $\sigma > 0$. For a.e. $x \in \Omega$, $r > 0$ define

$$\mathcal{U}_r^{\sigma, x} := \{z \in B_r(x) : |u(z) - u(x) - \nabla u(x)(z - x)| < \sigma |z - x|\} \quad (100)$$

and

$$\mathcal{D}_r^{\sigma, x} := \{z \in B_r(x) : |v(z) - v(x) - \nabla v(x)(z - x)| < \sigma |z - x|\}. \quad (101)$$

Now for a.e. $x \in \Omega$ there exists $\mu \in (0, w_x)$ such that for any $\phi \in S^{n-1}$ we can find $y_1 \in B_{\sigma\mu}(x) \cap \mathcal{U}_\mu^{\sigma, x} \cap \mathcal{D}_\mu^{\sigma, x}$ and $y_2 \in A(x, \frac{\mu}{2}, \mu) \cap \mathcal{U}_\mu^{\sigma, x} \cap \mathcal{D}_\mu^{\sigma, x}$ such that

$$\left| \left(\frac{y_2 - y_1}{|y_2 - y_1|} \right) - \phi \right| \leq \sigma^{\frac{1}{n-1}} \quad (102)$$

and for affine function L_{R_x} with $\nabla L_{R_x} = R_x \in SO(n)$

$$|u(y_i) - L_{R_x}(v(y_i))| \leq \sigma \mu \text{ for } i = 1, 2. \quad (103)$$

In addition

$$|B_r(z) \setminus \mathcal{U}_r^{\sigma, z}| \leq \sigma^{4n} r^n \text{ and } |B_r(z) \setminus \mathcal{D}_r^{\sigma, z}| \leq \sigma^{4n} r^n \text{ for all } r \in (0, \mu]. \quad (104)$$

Proof of Step 2. By Theorem 1.4 [Ge-Iw 99] we know u is a mapping of integrable and $\frac{\|\nabla u(z)\|^n}{\det(\nabla u(z))} \leq K(z)$ for a.e. $z \in \Omega$. So in particular $\det(\nabla u(z)) > 0$ for a.e. $z \in \Omega$. By Theorem 1, Section 6.1.1. [Ev-Ga 92] for a.e. $x \in \Omega$ we can find $\mu \in (0, w_x)$ such that (104) holds true and

$$\int_{B_r(x)} |\nabla u(x) - \nabla u(z)| dz \leq \sigma^{4n} \text{ and } \int_{B_r(x)} |\nabla v(x) - \nabla v(z)| dz \leq \sigma^{4n} \text{ for all } r \in (0, \mu] \quad (105)$$

Fix an x for which this is true and for which Step 1 holds. By Step 1 we can find $N_x \in \mathbb{N}$ such that

$$\int_{B_\mu(x)} |\nabla u_k - R_k \nabla v_k| dz \leq \sigma^{4n} \frac{\mu^n}{2} \text{ for all } k \geq N_x. \quad (106)$$

Passing to a subsequence (not relabeled) we have $R_k \rightarrow R_x$ as $k \rightarrow \infty$. As u_k, v_k are equibounded in L^1 this implies there exists $M_x \geq N_x$ such that

$$\int_{B_\mu(x)} |\nabla u_k - R_x \nabla v_k| dz \leq \sigma^{4n} \mu^n \text{ for all } k \geq M_x. \quad (107)$$

Now pick $k \geq M_x$ large enough so that

$$\|u_k - u\|_{L^1(\Omega)} \leq \sigma^{4n} \mu^{n+1}, \quad \|v_k - v\|_{L^1(\Omega)} \leq \sigma^{4n} \mu^{n+1}. \quad (108)$$

By Poincare inequality from (107) there exists affine function L_{R_x} with $\nabla L_{R_x} = R_x$ with

$$\int_{B_\mu(x)} |u_k - L_{R_x} \circ v_k| dz \leq c \sigma^{4n} \mu^{n+1}.$$

Hence by (108) we have

$$\int_{B_\mu(x)} |u - L_{R_x} \circ v| dz \leq c \sigma^{4n} \mu^{n+1}. \quad (109)$$

So let

$$\mathcal{I} := \{z \in B_\mu(x) : |u(z) - L_{R_x}(v(z))| \leq \sigma \mu\} \quad (110)$$

thus

$$|B_\mu(x) \setminus \mathcal{I}| \leq c \sigma^{4n-1} \mu^n. \quad (111)$$

Let $\mathcal{H} := \mathcal{I} \cap (\mathcal{U}_\mu^{\sigma, x} \cup \mathcal{D}_\mu^{\sigma, x})$. Since by (104) and (111)

$$|B_\mu(x) \setminus \mathcal{H}| \leq c \sigma^{4n-1} \mu^n. \quad (112)$$

Now let

$$\mathcal{E} := \left\{ y \in B_{\sigma\mu}(x) : \int \mathbb{1}_{B_\mu(x) \setminus \mathcal{H}}(z) |z - y|^{-n+1} dz \leq \sigma^2 \mu \right\} \quad (113)$$

Now

$$\begin{aligned} & \int_{B_{\sigma\mu}(x)} \int \mathbb{1}_{B_\mu(x) \setminus \mathcal{H}}(z) |z - y|^{-n+1} dz dy \\ &= \int \mathbb{1}_{B_\mu(x) \setminus \mathcal{H}}(z) \left(\int_{B_{\sigma\mu}(x)} |z - y|^{-n+1} dy \right) dz \\ &= c \sigma \mu \int \mathbb{1}_{B_\mu(x) \setminus \mathcal{H}}(z) dz \\ &\stackrel{(112)}{\leq} c \sigma^{4n} \mu^{n+1}. \end{aligned} \quad (114)$$

Hence $|B_{\sigma\mu}(x) \setminus \mathcal{E}| \sigma^2 \mu \stackrel{(113), (114)}{\leq} c \sigma^{4n} \mu^{n+1}$ so

$$|B_{\sigma\mu}(x) \setminus \mathcal{E}| \leq c \sigma^{4n-2} \mu^n. \quad (115)$$

By (112), (115) it is clear $|B_{\sigma\mu}(x) \cap (\mathcal{E} \cup \mathcal{H})| > 0$ so pick $y_1 \in \mathcal{E} \cap \mathcal{H} \cap B_{\sigma\mu}(x)$, define $I_{y_1}^\theta := \{y_1 + \mathbb{R}_+ \theta\}$. So by the Co-area formula into S^{n-1} (see for example [Je-Lor 08] Lemma 14), by definition of \mathcal{E} (recall (113))

$$\int_{\theta \in S^{n-1}} \int_{I_{y_1}^\theta} \mathbb{1}_{B_\mu(x) \setminus \mathcal{H}} dH^1 z dH^{n-1} \theta \leq \sigma^2 \mu. \quad (116)$$

So let

$$\Psi_{y_1} := \left\{ \theta \in S^{n-1} : \int_{I_{y_1}^\theta} \mathbb{1}_{B_\mu(x) \setminus \mathcal{H}} dH^1 z \leq \sigma \mu \right\}$$

thus by (116), $H^{n-1}(S^{n-1} \setminus \Psi_{y_1}) \leq \sigma$. Thus we can find $\psi \in \Psi_{y_1}$ such that $|\psi - \theta| \leq c \sigma^{\frac{1}{n-1}}$. Thus we can find $y_2 \in I_{y_1}^\psi \cap A(x, \frac{\mu}{2}, \mu) \cap \mathcal{H}$. Since $y_1, y_2 \in \mathcal{H} \subset \mathcal{I}$ by definition (110) we know they satisfy

(103). Since $\frac{y_2 - y_1}{|y_2 - y_1|} = \psi$ it is clear that (102) is satisfied. This completes the proof of Step 2.

Step 3. We will show that for a.e. $x \in \Omega$ there exists $R_x \in SO(n)$ such that

$$R_x \nabla v(x) = \nabla u(x). \quad (117)$$

Proof of Step 3. Let $x \in \Omega$ be one of the a.e. points x such that the conclusion of Step 2 hold true. Let $\gamma > 0$ and set

$$\sigma = \left(\frac{\gamma}{|\nabla u(x)| + |\nabla v(x)| + 1} \right)^{n-1}. \quad (118)$$

By Step 3 we can find and points $y_1 \in B_{\sigma\mu}(x)$, $y_2 \in A(x, \frac{\mu}{2}, \mu)$ such that (102), (103) are satisfied. So since $y_i \in \mathcal{U}_\mu^{\sigma, x}$

$$|u(y_i) - u(x) - (y_i - x) \cdot \nabla u(x)| < \sigma |y_i - x| < \sigma\mu \text{ for } i = 1, 2$$

taking one inequality away from another

$$|(u(y_1) - u(y_2)) - \nabla u(x)(y_1 - y_2)| < 2\sigma\mu. \quad (119)$$

And in the same way

$$|(v(y_1) - v(y_2)) - \nabla v(x)(y_1 - y_2)| < 2\sigma\mu. \quad (120)$$

Applying (103) to (119) we have $|R_x(v(y_1) - v(y_2)) - \nabla u(x)(y_1 - y_2)| < 4\sigma\mu$ and putting this together with (120) we $|\nabla v(x)(y_1 - y_2) - R_x^{-1}\nabla u(x)(y_1 - y_2)| \leq 6\sigma\mu$ since $|y_1 - y_2| > \frac{\mu}{2}$ so

$$\left| \nabla v(x) \frac{(y_2 - y_1)}{|y_2 - y_1|} - R_x^{-1} \nabla u(x) \frac{(y_2 - y_1)}{|y_2 - y_1|} \right| \leq 12\sigma. \quad (121)$$

Thus $|(\nabla v(x) - R_x^{-1}\nabla u(x))\phi| \stackrel{(121)}{\leq} 12\sigma + |(\nabla v(x) - R_x^{-1}\nabla u(x))| \left| \phi - \frac{y_2 - y_1}{|y_2 - y_1|} \right| \stackrel{(102), (121), (118)}{\leq} 14\gamma$. Now as γ is arbitrary this implies $R_x \nabla v(x) = \nabla u(x)$. This completes the proof of the lemma. \square

3.1. Proof of Theorem 3 completed. Let $v \in W^{1,p}(\Omega : \mathbb{R}^n)$ and $u \in W^{1,q}(\Omega : \mathbb{R}^n)$ be the weak limit of v_k, u_k . We know by Lemma 4 $S(\nabla u) = S(\nabla v)$ for a.e. $x \in \Omega$. So we can apply Theorem 1 and thus there exists $R \in SO(n)$ such that

$$\nabla v(z) = R \nabla u(z) \text{ for a.e. } z \in \Omega. \quad (122)$$

Since $\det(\nabla u(z)) > 0$ for a.e. $z \in \Omega$ so $\int_\Omega \det(\nabla u) dz \leq \int_\Omega \|\nabla u\|^n dz \leq C$. For any $\gamma > 0$ let

$$\mathcal{D}_\gamma := \left\{ z \in \Omega : \det(\nabla u(z)) < \gamma^{\frac{1}{100}} \right\} \quad (123)$$

and let

$$\mathcal{U}_\gamma := \left\{ z \in \Omega : |\nabla u(z)| > \gamma^{-\frac{1}{100}} \right\}. \quad (124)$$

Note $|\mathcal{U}_\gamma| \rightarrow 0$ and $|\mathcal{D}_\gamma| \rightarrow 0$ as $\gamma \rightarrow 0$. Let $\delta \in (0, 1)$. Define

$$\mathcal{O}_\delta := \left\{ x \in \Omega : K(x) \geq \delta^{-\frac{1}{100}} \right\}, \quad (125)$$

note

$$|\mathcal{O}_\delta| \leq c\delta^{\frac{1}{100}}. \quad (126)$$

Let $\epsilon \in (0, \delta)$ be small enough so that

$$\int_{\mathcal{U}_\epsilon} \det(\nabla u) dz \leq \delta^n. \quad (127)$$

For a.e. $x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta)$ let L_x be the affine map defined by $L_x(z) = u(x) + \nabla u(x)z$. Recall from Lemma 3 we defined $\Lambda(A) := \inf_{v \in S^{n-1}} |Av|$ and from (46) we know

$$\frac{\det(\nabla u(x))}{K^{n-1}(x)} \leq (\Lambda(\nabla u(x)))^n \text{ for a.e. } x \in \Omega. \quad (128)$$

Since $x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta)$, $K(x) < \delta^{-\frac{1}{100}}$ and $\det(\nabla u(x)) > \delta^{\frac{1}{100}}$ so $\delta^{\frac{1}{100}} \stackrel{(128)}{\leq} \Lambda(\nabla u(x))$.

Thus

$$B_{\delta^{\frac{1}{100}}h}(u(x)) \subset L_x(B_h(x)) \text{ for all } h > 0. \quad (129)$$

Now by uniform continuity of u , approximate differentiability of u and approximate continuity of $\det(\nabla u)$ for a.e. $x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta)$ there exists $p_x > 0$ such that

$$H_d(u(B_h(x)), L_x(B_h(x))) \leq \epsilon \delta^{\frac{1}{100}}h \text{ for any } h \in (0, p_x) \quad (130)$$

and

$$|u(B_h(x))| - \Gamma(n) \det(\nabla u(x))h^n \leq \epsilon h^n \text{ for any } h \in (0, p_x). \quad (131)$$

Note by (129), (130) we have that

$$B_{8^{-1}\delta^{\frac{1}{100}}h}(u(x)) \subset u(B_{\frac{h}{4}}(x)) \text{ for any } h \in (0, p_x), x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta) \quad (132)$$

Now let

$$\mathcal{H}_\rho := \{x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta) : p_x < \rho\}. \quad (133)$$

Let us choose ρ_0 be such that

$$|\mathcal{H}_{\rho_0}| < \delta. \quad (134)$$

Note also that by Lebesgue density theorem for a.e. $x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0})$ the ratio

$$\frac{|B_r(x) \cap (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0})|}{r^n}$$

can be arbitrarily small. For each x we need to find the $q_x > 0$ such that for all $r \in (0, q_x)$ the ratio is less than a small constant depending on δ . Rather than introduce more notation to signify this small quantity then later take it to be less than the constant we need, we find q_x that has the exact property we need in terms of δ . So for a.e. $x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0})$ there exists $q_x > 0$ such that

$$|B_r(x) \cap (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0})| \leq \frac{C_0^n}{32^n} r^n \exp(-64^n n \delta^{-\frac{101n}{100}}) \text{ for } r \in (0, q_x). \quad (135)$$

For $\rho > 0$ let

$$\Theta_\rho := \{x \in \Omega \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0}) : q_x < \rho\}. \quad (136)$$

We can find $\rho_1 > 0$ such that

$$|\Theta_{\rho_1}| \leq \delta. \quad (137)$$

By Lemma 3 for a.e. $x \in \Omega$ there exists $w_x \in (0, q_x)$ and $N_x \in \mathbb{N}$ such that for $u|_{B_{w_x}(x)}$ and $u_k|_{B_{w_x}(x)}$ are injective for any $k \geq N_x$. We can find $\tau > 0$ and $M_0 \in \mathbb{N}$ such that

$$|\{x \in \Omega : w_x < \tau\}| < \epsilon \text{ and } |\{x \in \Omega : N_x > M_0\}| < \epsilon. \quad (138)$$

Let $\mathcal{E}_\epsilon := \{x \in \Omega : w_x < \tau\} \cup \{x \in \Omega : N_x > M_0\}$. Now recall the notation $N_h(\cdot)$ (see (18)). Define

$$\Pi := \Omega \setminus (\mathcal{O}_\delta \cup \mathcal{D}_\delta \cup \mathcal{H}_{\rho_0} \cup \mathcal{U}_\delta \cup \mathcal{E}_\epsilon \cup \Theta_{\rho_1} \cup N_\delta(\partial\Omega)). \quad (139)$$

Step 1. Let $\eta = \frac{1}{2} \min\{\tau, \rho_0\}$. Since $\{B_\eta(x) : x \in \Pi\}$ is a cover of Π , again by Theorem 2.7 [Ma 95] we can find a collection $\{B_{\frac{\eta}{2}}(x_1), B_{\frac{\eta}{2}}(x_2), \dots, B_{\frac{\eta}{2}}(x_m)\}$ such that

$$\Pi \subset \bigcup_{i=1}^m B_{\frac{\eta}{2}}(x_i) \quad (140)$$

and

$$\sum_{k=1}^m \mathbb{1}_{B_{2\eta}(x_i)} \leq c. \quad (141)$$

Let γ be some small positive number we decide on later. Let $M_1 > M_0$ be such that

$$\int_{\Omega} |S(\nabla u_q) - S(\nabla v_q)|^p + |1 - \text{sgn}(\det(\nabla v_q))| dz \leq \gamma \text{ for all } q > M_1.$$

Fix $k > M_1$, define

$$B_0^k := \left\{ i \in \{1, 2, \dots, m\} : \int_{B_{\eta}(x_i)} |\nabla u_k|^n dz \geq \delta^{-n} \right\}, \quad (142)$$

$$B_1^k := \left\{ i \in \{1, 2, \dots, m\} : \int_{B_{\eta}(x_i)} |S(\nabla u_k) - S(\nabla v_k)|^p + |1 - \text{sgn}(\det(\nabla v_k))| dz \geq \sqrt{\gamma} \right\}, \quad (143)$$

$$B_2 := \left\{ i \in \{1, 2, \dots, m\} : |B_{\eta}(x_i) \cap \mathcal{D}_{\delta}| \geq \Gamma(n) 2^{n-1} \eta^n \right\}, \quad (144)$$

$$B_3 := \left\{ i \in \{1, 2, \dots, m\} : \int_{B_{\eta}(x_i) \cap \mathcal{U}_{\epsilon}} \det(\nabla u) dz \geq \delta^{\frac{n}{4}} \int_{B_{\eta}(x_i)} \det(\nabla u) dz \right\}. \quad (145)$$

We will show

$$\text{Card} \left(B_0^k \cup B_1^k \cup B_2 \cup B_3 \right) \leq \left(c \sqrt{\gamma} + c \delta^{\frac{n}{2}} + c |\mathcal{D}_{\delta}| \right) \eta^{-n} \quad (146)$$

and any $i \in \{1, 2, \dots, m\} \setminus (B_0^k \cup B_1^k \cup B_2 \cup B_3)$ there exists $R_i^k \in SO(n)$ such that

$$\int_{B_{\frac{\eta}{2}}(x_i)} \left| \nabla v_k - R_i^k \nabla u_k \right| dz \leq c \epsilon^{2n} \quad (147)$$

and (recalling $R \in SO(n)$ satisfies (122)) for affine maps $l_R, l_{R_i^k}$ with $\nabla l_R = R, \nabla l_{R_i^k} = R_i^k$

$$\eta^{-n} \int_{u(B_{\eta}(x_i) \setminus \mathcal{U}_{\epsilon})} \left| l_R(z) - l_{R_i^k}(z) \right| \det(\nabla u^{-1}(z)) dz \leq c \eta \epsilon^{2n}. \quad (148)$$

Proof of Step 1. Note that since u_k is an equibounded sequence in $W^{1,n}$, so

$$c \delta^{-n} \eta^n \text{Card} \left(B_0^k \right) \leq \int_{B_{\eta}(x_i)} |\nabla u_k|^n dz \leq c$$

thus $\text{Card} \left(B_0^k \right) \leq \frac{c \delta^n}{\eta^n}$. It is also clear $\sqrt{\gamma} \eta^n \text{Card} \left(B_1^k \right) \leq c \gamma$ and thus $\text{Card} \left(B_1^k \right) \leq \sqrt{\gamma} \eta^{-n}$. Also

$$\text{Card} \left(B_2 \right) \eta^n \leq c |\mathcal{D}_{\delta}|$$

so $\text{Card} \left(B_2 \right) \leq c |\mathcal{D}_{\delta}| \eta^{-n}$. And

$$\begin{aligned} c \delta^n &\stackrel{(127),(141)}{\geq} \sum_{i \in B_3 \setminus B_2} \int_{B_{2\eta}(x_i) \cap \mathcal{U}_{\epsilon}} \det(\nabla u) dx \\ &\stackrel{(145)}{\geq} c \delta^{\frac{n}{4}} \sum_{i \in B_3 \setminus B_2} \int_{B_{2\eta}(x_i)} \det(\nabla u) dx \\ &\stackrel{(144),(123)}{\geq} c \delta^{\frac{n}{2}} \eta^n \text{Card} \left(B_3 \setminus B_2 \right). \end{aligned}$$

Thus $\text{Card} \left(B_3 \setminus B_2 \right) \leq c \delta^{\frac{n}{2}} \eta^{-n}$. Now by definition of Θ_{ρ_1} (see (136), (135)), since we know from Step 1 $\eta < \rho_0$ and $x_1, x_2, \dots, x_m \notin \Theta_{\rho_0}$ we have that for each $i = 1, 2, \dots, m$

$$|B_{\eta}(x_i) \cap (\mathcal{D}_{\delta} \cup \mathcal{O}_{\delta} \cup \mathcal{H}_{\rho_0})| \leq \frac{C_0^n}{32^n} \eta^n \exp(-64^n n \delta^{-\frac{101n}{100}}). \quad (149)$$

And by (132), (133)

$$B_{8^{-1}\delta^{\frac{1}{100}}h}(u(z)) \subset u(B_{\frac{h}{4}}(z)) \text{ for each } z \in B_{\frac{\eta}{2}}(x_i) \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0}), h \in (0, \frac{\eta}{2}), i = 1, 2, \dots, m. \quad (150)$$

Recalling from (91) we know equicontinuity of the sequence u_k on a compact subset of Ω and hence uniform convergence of u_k . So let $Q \in \mathbb{N}$ be such that

$$\|u - u_k\|_{L^\infty(\Omega \setminus N_{\frac{\delta}{2}}(\partial\Omega))} < 16^{-3}\eta\delta^{\frac{1}{100}}C_0 \exp(-64^n\delta^{-\frac{101n}{100}}) \text{ for all } k \geq Q. \quad (151)$$

So (recalling that $B_{\frac{\eta}{2}}(x_i) \subset N_{\frac{\eta}{2}}(\Pi) \stackrel{(139)}{\subset} \Omega \setminus N_{\frac{\delta}{2}}(\partial\Omega)$)

$$u(B_{\frac{h}{4}}(z)) \setminus N_{16^{-2}\eta\delta^{\frac{1}{100}}C_0 \exp(-64^n\delta^{-\frac{101n}{100}})}(\partial u(B_{\frac{h}{4}}(z))) \stackrel{(151)}{\subset} u_k(B_{\frac{h}{4}}(z)) \text{ for any } k \geq Q. \quad (152)$$

Thus for any $z \in B_{\frac{\eta}{2}}(x_i) \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0})$, $h \in [\frac{C_0}{16}\eta \exp(-64^n\delta^{-\frac{101n}{100}}), \frac{\eta}{2}]$ we have

$$\begin{aligned} d(B_{16^{-1}\delta^{\frac{1}{100}}h}(u(z)), \partial u(B_{\frac{h}{4}}(z))) &\stackrel{(150)}{\geq} 16^{-1}\delta^{\frac{1}{100}}h \\ &\geq 16^{-2}\delta^{\frac{1}{100}}C_0\eta \exp(-64^n\delta^{-\frac{101n}{100}}) \end{aligned} \quad (153)$$

and thus

$$\begin{aligned} B_{16^{-1}\delta^{\frac{1}{100}}h}(u(z)) &\subset u(B_{\frac{h}{4}}(z)) \setminus N_{16^{-2}\eta\delta^{\frac{1}{100}}C_0\eta \exp(-64^n\delta^{-\frac{101n}{100}})}(\partial u(B_{\frac{h}{4}}(z))) \\ &\stackrel{(152)}{\subset} u_k(B_{\frac{h}{4}}(z)) \text{ for } k \geq Q. \end{aligned} \quad (154)$$

Hence as $\|u - u_k\|_{L^\infty(\Omega \setminus N_{\frac{\delta}{2}}(\partial\Omega))} \stackrel{(151)}{\leq} 16^{-2}\delta^{\frac{1}{100}}h$ we have

$$\begin{aligned} B_{\frac{15}{16^2}\delta^{\frac{1}{100}}h}(u_k(z)) &\stackrel{(154)}{\subset} u_k(B_{\frac{h}{4}}(z)) \text{ for } z \in B_{\frac{\eta}{2}}(x_i) \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0}), \\ h &\in [\frac{C_0}{16}\eta \exp(-64^n\delta^{-\frac{101n}{100}}), \frac{\eta}{2}] \text{ and } k \geq Q. \end{aligned} \quad (155)$$

If $i \in \{1, 2, \dots, m\} \setminus (B_1^k \cup B_0^k)$ we can define $\Xi := B_{\frac{\eta}{2}}(x_i) \setminus (\mathcal{D}_\delta \cup \mathcal{O}_\delta \cup \mathcal{H}_{\rho_0})$ and define

$$\mathcal{E} = \frac{15}{16^2}\delta^{\frac{1}{100}} \text{ and } A = \delta^{-n}2^n \quad (156)$$

and notice that

$$\frac{C_0}{8} \frac{\eta}{2} \exp\left(-\frac{A}{\mathcal{E}^n}\right) > \frac{C_0}{16} \eta \exp(-64^n\delta^{-\frac{101n}{100}}) \text{ and } \frac{C_0^n}{16^n} \frac{\eta^n}{2^n} \exp\left(-\frac{nA}{\mathcal{E}^n}\right) > \frac{C_0^n}{32^n} \eta^n \exp(-64^n n \delta^{-\frac{101n}{100}})$$

thus by (155), (149) hypotheses for $(r$ taken to be $\frac{\eta}{2})$ (66) and (65) is satisfied. So we can apply Lemma 2 (taking A, \mathcal{E} defined by (156) and $\epsilon = \sqrt{\gamma}$). In addition in view of (142), (143) hypotheses (62), (63) and (64) are satisfied and there exists $R_i^k \in SO(n)$ such that

$$\int_{B_{\frac{\eta}{4}}(x_i)} |\nabla v_k - R_i^k \nabla u_k| dx \leq c\delta^{-4n^2} \exp\left(40^{n+1}5n^3\delta^{-\frac{101n}{100}}\right) \left(\ln\left(2 + \frac{\gamma^{-\frac{1}{8}}}{2^n}\right)\right)^{-\frac{1}{64n^2}} \quad (157)$$

so assuming γ was chosen small enough (147) is established.

By Poincare's inequality there exists affine map $l_{R_i^k}$ with $\nabla l_{R_i^k} = R_i^k$, so

$$\int_{B_{\frac{\eta}{4}}(x_i)} |v_k - l_{R_i^k} \circ u_k| dx \leq c\eta\epsilon^{2n}. \quad (158)$$

Now as $v_k \xrightarrow{L^1(\Omega)} v$ and $u_k \xrightarrow{L^1(\Omega)} u$ so assuming k is large enough we have

$$\int_{B_{\frac{\eta}{4}}(x_i)} |v_k - v| dx \leq c\eta\epsilon^{2n} \text{ and } \int_{B_{\frac{\eta}{4}}(x_i)} |u_k - u| dx \leq c\eta\epsilon^{2n}$$

putting this together with (158) we have

$$\int_{B_{\frac{\eta}{4}}(x_i)} |v - l_{R_i^k} \circ u| dx \leq c\eta\epsilon^{2n}. \quad (159)$$

Since $\nabla v = R\nabla u$ for some affine map l_R with $\nabla l_R = R$ we have $v = l_R \circ u$ on Ω . So putting this together with (159) we have

$$\int_{B_{\frac{\eta}{4}}(x_i)} |l_R \circ u - l_{R_i^k} \circ u| dx \leq c\eta\epsilon^{2n}. \quad (160)$$

Thus if $i \notin (B_0^k \cup B_1^k \cup B_2 \cup B_3)$

$$\begin{aligned} c\eta\epsilon^{2n} &\geq \int_{B_{\frac{\eta}{4}}(x_i) \setminus \mathcal{U}_\epsilon} |l_R(u(z)) - l_{R_i^k}(u(z))| \det((\nabla u(u^{-1}(u(z))))^{-1}) \det(\nabla u(z)) dz \\ &= c\eta^{-n} \int_{u(B_{\frac{\eta}{4}}(x_i) \setminus \mathcal{U}_\epsilon)} |l_R(z) - l_{R_i^k}(z)| \det(\nabla u^{-1}(z)) dz. \end{aligned} \quad (161)$$

This completes the proof of Step 1.

Step 2. We will show that for any $k > M_1$

$$|R - R_i^k| \leq c\epsilon^{\frac{n}{4}} \text{ for any } i \in \{1, 2, \dots, M\} \setminus (B_0^k \cup B_1^k \cup B_2 \cup B_3). \quad (162)$$

Proof of Step 2. Now since $i \notin B_3$, (see (145) for the definition) and we chose $x_i \notin \mathcal{U}_\delta$ (recall (124) for the definition)

$$\begin{aligned} |u(\mathcal{U}_\epsilon \cap B_\eta(x_i))| &\stackrel{(145)}{<} \delta^{\frac{n}{4}} |u(B_\eta(x_i))| \\ &\stackrel{(131), (124)}{\leq} c\delta^{\frac{n}{4}} \delta^{-\frac{n}{100}} \eta^n \\ &\leq c\delta^{\frac{24n}{100}} \eta^n. \end{aligned} \quad (163)$$

So as $x_i \in \Pi$ thus $x_i \notin \stackrel{(139)}{(\mathcal{D}_\delta \cup \mathcal{O}_\delta)}$ (recall definition (123)) and so $\det(\nabla u(x_i)) \geq \delta^{\frac{1}{100}}$ and hence

$$\begin{aligned} |u(B_\eta(x_i) \setminus \mathcal{U}_\epsilon)| &\stackrel{(131), (163)}{\geq} c\delta^{\frac{1}{100}} \eta^n - c\epsilon \eta^n \\ &\geq c\delta^{\frac{1}{100}} \eta^n. \end{aligned} \quad (164)$$

Now by (132), (133) (since $x_i \notin \mathcal{H}_{\rho_0}$ and $\eta \leq \frac{\rho_0}{2}$) we have

$$B_{8^{-1}\delta^{\frac{1}{100}}\eta}(u(x_i)) \subset u\left(B_{\frac{\eta}{4}}(x_i)\right). \quad (165)$$

So define

$$\begin{aligned} A &:= B_{\frac{1}{\delta^{\frac{100}{64}}\eta}} \left(u(x_i) + e_1 \frac{\delta^{\frac{1}{100}}\eta}{16} \right) \setminus u(\mathcal{U}_\epsilon \cap B_\eta(x_i)) \\ \text{and } B &:= B_{\frac{1}{\delta^{\frac{100}{64}}\eta}} \left(u(x_i) - e_1 \frac{\delta^{\frac{1}{100}}\eta}{16} \right) \setminus u(\mathcal{U}_\epsilon \cap B_\eta(x_i)). \end{aligned} \quad (166)$$

By (166), (165) and the fact u is injective on $B_\eta(x_i)$ (recall (138), (139))

$$A \cup B \subset u(B_\eta(x_i) \setminus \mathcal{U}_\epsilon) \quad (167)$$

and note

$$\text{dist}(A, B) > \frac{\delta^{\frac{1}{100}}\eta}{64}. \quad (168)$$

Now

$$\begin{aligned} |A| &\stackrel{(163),(166)}{\geq} \frac{\delta^{\frac{n}{100}}\eta^n}{64^n} - c\delta^{\frac{24n}{100}}\eta^n \\ &\geq c\delta^{\frac{n}{100}}\eta^n. \end{aligned} \quad (169)$$

In exactly the same way $|B| \geq c\delta^{\frac{n}{100}}\eta^n$. Now note

$$\begin{aligned} \eta^{-n} \epsilon^{\frac{n}{100}} \int_{u(B_\eta(x_i) \setminus \mathcal{U}_\epsilon)} |l_R(z) - l_{R_i}(z)| dz \\ &\stackrel{(124)}{\leq} \eta^{-n} \int_{u(B_\eta(x_i) \setminus \mathcal{U}_\epsilon)} |l_R(z) - l_{R_i}(z)| \det(\nabla u(u^{-1}(z)))^{-1} dz \\ &\stackrel{(148)}{\leq} c\eta\epsilon^{2n}. \end{aligned} \quad (170)$$

Let

$$U_A := \left\{ z \in A : |l_R(z) - l_{R_i^k}(z)| > \eta\epsilon^{\frac{n}{2}} \right\}. \quad (171)$$

Notice

$$\begin{aligned} \eta\epsilon^{\frac{n}{2}} |U_A| &\leq \int_A |l_R(z) - l_{R_i^k}(z)| dz \\ &\stackrel{(170)}{\leq} c\eta^{n+1} \epsilon^{\frac{199n}{100}}. \end{aligned}$$

So $|U_A| \leq c\eta^n\epsilon$, since $\epsilon \ll \delta$, from (169) $|A \setminus U_A| > 0$ and we can pick $x_A \in A \setminus U_A$. In exactly the same way $x_B \in B \setminus U_B$. So $|l_R(x_A) - l_{R_i^k}(x_A)| \leq \eta\epsilon^{\frac{n}{2}}$ and $|l_R(x_B) - l_{R_i^k}(x_B)| \leq c\eta\epsilon^{\frac{n}{2}}$. Now $l_R(z) = Rz + \alpha_R$ and $l_{R_i^k}(z) = R_i^k z + \alpha_{R_i}$ for some $\alpha_R, \alpha_{R_i} \in \mathbb{R}^n$, so

$$|(R - R_i^k)x_A + (\alpha_R - \alpha_{R_i^k})| \leq c\eta\epsilon^{\frac{n}{2}}$$

and

$$|(R - R_i^k)x_B + (\alpha_R - \alpha_{R_i^k})| \leq c\eta\epsilon^{\frac{n}{2}}.$$

Now taking one away from another $|(R - R_i^k)(x_A - x_B)| \leq c\eta\epsilon^{\frac{n}{2}}$. Note $|x_A - x_B| \stackrel{(168)}{\geq} \frac{\delta^{\frac{1}{100}}\eta}{64}$ so

$$\left| (R - R_i^k) \frac{(x_A - x_B)}{|x_A - x_B|} \right| \leq c\delta^{-\frac{1}{100}}\epsilon^{\frac{n}{2}} \leq c\epsilon^{\frac{n}{4}}$$

Therefor $|R - R_i^k| \leq c\epsilon^{\frac{n}{4}}$, this completes the proof of Step 2.

Final step of Proof of Theorem 3.

Let $k > M_1$. For any $i \in \{1, 2, \dots, m\} \setminus (B_0^k \cup B_1^k \cup B_2 \cup B_3)$

$$\begin{aligned}
 \int_{B_{\frac{\eta}{2}}(x_i)} |\nabla v_k - R \nabla u_k| dz &\leq \int_{B_{\frac{\eta}{2}}(x_i)} |\nabla v_k - R_i^k \nabla u_k| + |R_i^k \nabla u_k - R \nabla u_k| dz \\
 &\stackrel{(147)}{\leq} c\epsilon^{2n} + |R_i^k - R| \int_{B_{\frac{\eta}{2}}(x_i)} |\nabla u_k| dz \\
 &\stackrel{(162)}{\leq} c\epsilon^{2n} + c\epsilon^{\frac{n}{4}} \int_{B_{\frac{\eta}{2}}(x_i)} |\nabla u_k| dz \\
 &\stackrel{(142)}{\leq} c\delta^{-1} \epsilon^{\frac{n}{4}} \leq c\epsilon^{\frac{n}{8}}.
 \end{aligned} \tag{172}$$

Let

$$\Pi' := \Pi \cap \left(\bigcup_{\{1, 2, \dots, m\} \setminus (B_0^k \cup B_1^k \cup B_2 \cup B_3)} B_{\frac{\eta}{2}}(x_i) \right) \tag{173}$$

so

$$\begin{aligned}
 |\Pi \setminus \Pi'| &\stackrel{(173)}{\leq} c\eta^n \text{Card} \left(B_0^k \cup B_1^k \cup B_2 \cup B_3 \right) \\
 &\stackrel{(146)}{\leq} c\sqrt{\gamma} + c\delta^{\frac{n}{2}} + c|\mathcal{D}_\delta|.
 \end{aligned} \tag{174}$$

Now recall from definition of Π (139) we have that

$$\begin{aligned}
 |\Omega \setminus \Pi| &\leq |\mathcal{O}_\delta| + |\mathcal{U}_\delta| + |\mathcal{D}_\delta| + |\mathcal{E}_\epsilon| + |\Theta_{\rho_1}| + |\mathcal{H}_{\rho_0}| + |N_\delta(\partial\Omega)| \\
 &\stackrel{(126), (138), (137), (134)}{\leq} c\delta^{\frac{1}{100}} + |\mathcal{U}_\delta| + |\mathcal{D}_\delta| + c\epsilon.
 \end{aligned} \tag{175}$$

So putting (174), (175) together we have

$$|\Omega \setminus \Pi'| \leq c\delta^{\frac{1}{100}} + |\mathcal{U}_\delta| + |\mathcal{D}_\delta| + c\epsilon + c\sqrt{\gamma}. \tag{176}$$

So from (172) and definition (173) we have

$$\begin{aligned}
 \int_{\Pi'} |\nabla v_k - R \nabla u_k| dz &\leq \sum_{i \in \{1, 2, \dots, m\} \setminus (B_0^k \cup B_1^k \cup B_2 \cup B_3)} \int_{B_{\frac{\eta}{2}}(x_i)} |\nabla v_k - R \nabla u_k| dz \\
 &\leq c\epsilon^{\frac{n}{8}}.
 \end{aligned} \tag{177}$$

To simplify notation let $\varsigma_k = \int_\Omega |S(\nabla v_k) - S(\nabla u_k)| dz$

$$\begin{aligned}
 \left| \int_\Omega |\nabla v_k| dz - \int_\Omega |\nabla u_k| dz \right| &\leq \int_\Omega ||\nabla v_k| - |\nabla u_k|| dz \\
 &\leq c \int_\Omega ||S(\nabla v_k)| - |S(\nabla u_k)|| dz \\
 &\leq c \int_\Omega |S(\nabla v_k) - S(\nabla u_k)| dz \\
 &\leq c\varsigma_k.
 \end{aligned} \tag{178}$$

Note also that

$$\begin{aligned} \int_{\Omega \setminus \Pi'} |\nabla u_k| dz &\leq \left(\int_{\Omega} |\nabla u_k|^n dz \right)^{\frac{1}{n}} |\Omega \setminus \Pi'|^{\frac{n-1}{n}} \\ &\stackrel{(176)}{\leq} c \left(\delta^{\frac{1}{100}} + |\mathcal{U}_{\delta}| + |\mathcal{D}_{\delta}| + \sqrt{\gamma} + \epsilon \right)^{\frac{n-1}{n}}. \end{aligned} \quad (179)$$

And

$$\begin{aligned} \int_{\Omega \setminus \Pi'} |\nabla v_k| dz &\stackrel{(178)}{\leq} \int_{\Omega \setminus \Pi'} |\nabla u_k| dz + c\zeta_k \\ &\stackrel{(179)}{\leq} c \left(\delta^{\frac{1}{100}} + |\mathcal{U}_{\delta}| + |\mathcal{D}_{\delta}| + \sqrt{\gamma} + \epsilon \right)^{\frac{n-1}{n}} + c\zeta_k. \end{aligned} \quad (180)$$

Thus

$$\int_{\Omega \setminus \Pi'} |\nabla v_k - R \nabla u_k| dz \stackrel{(179), (180)}{\leq} c \left(\delta^{\frac{1}{100}} + |\mathcal{U}_{\delta}| + |\mathcal{D}_{\delta}| + \sqrt{\gamma} + \epsilon \right)^{\frac{n-1}{n}} + c\zeta_k. \quad (181)$$

Putting this together with (177) we have

$$\begin{aligned} \int_{\Omega} |\nabla v_k - R \nabla u_k| dz \\ \leq c \left(\delta^{\frac{1}{100}} + |\mathcal{U}_{\delta}| + |\mathcal{D}_{\delta}| + \sqrt{\gamma} + \epsilon^{\frac{1}{8}} \right)^{\frac{n-1}{n}} + c\zeta_k \text{ for all } k > M_1. \end{aligned}$$

Now recall $\epsilon \ll \delta$ are $\gamma \ll \epsilon$ and δ was chosen arbitrarily. So we have established (6). \square

4. COUNTER EXAMPLE

Example 1. Let $Q_1 := \{z : |z|_{\infty} < 1\}$. Define

$$u(x_1, x_2, \dots, x_n) := \begin{cases} (x_1, x_2 x_1, x_3, \dots, x_n) & \text{for } x_1 > 0 \\ (x_1, -x_2 x_1, x_3, \dots, x_n) & \text{for } x_1 \leq 0 \end{cases}$$

and for some $\theta \in (0, 2\pi)$

$$v(x_1, x_2, \dots, x_n) := \begin{cases} (x_1 \cos \theta - x_1 x_2 \sin \theta, x_1 \sin \theta + x_1 x_2 \cos \theta, x_3, \dots, x_n) & \text{for } x_1 > 0 \\ (x_1, -x_2 x_1, x_3, \dots, x_n) & \text{for } x_1 \leq 0 \end{cases}$$

Note that for $x_1 \leq 0$

$$\nabla u(x) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -x_2 & -x_1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

And for $x_1 > 0$

$$\begin{aligned} \nabla v(x) &= \begin{pmatrix} \cos \theta - x_2 \sin \theta & -x_1 \sin \theta & 0 & \dots & 0 \\ \sin \theta + x_2 \cos \theta & x_1 \cos \theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 & \dots & 0 \\ \sin \theta & \cos \theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ x_2 & x_1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \end{aligned}$$

Since $\nabla u(x) = \nabla v(x)$ for $x_1 \leq 0$ it is clear there is no R such that $\nabla v(x) = R\nabla u(x)$ for $x \in Q_1$. Now note that $\det(\nabla u(x)) = x_1$ for all $x \in Q$ and $|\nabla u(x)|^n = ((n-1) + x_2^2 + x_1^2)^{\frac{n}{2}}$ so defining $K(x) := |\nabla u(x)|^n / \det(\nabla u(x)) = x_1^{-1} ((n-1) + x_2^2 + x_1^2)^{\frac{n}{2}}$. So it is clear that $\int_{Q_1} K(z) dz = \infty$ and thus it follows that Theorems 1 and 3 are optimal for $n = 2$.

5. ON THE QUESTION OF SHARPNESS OF THEOREM 1 AND THEOREM 3

As mentioned the only known way of constructing a counter examples to Theorems 1 and 3 is to take a function that squeezes down a domain into a shape whose interior consists of two disjoint pieces. In three dimensions in analogy with Example 1 of Section 4 we could consider squeezing the center of a cube to a line, in effect doing the squeezing only in the x and z variables. However in this case the calculations reduce to those of the two dimensional situation and it can be shown that for a wide class of mappings, squeezing down the center to a line implies that the mapping fails to have L^1 integrable dilatation.

A more promising approach might be to consider mappings that squeeze down the center of a cylinder to a point. However Proposition 1 below will show, such examples (if they exist) can not be easily constructed.

Let $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be a rotation around the z -axis. We say Ω is axially symmetric

if $R_\theta \Omega = \Omega$ for every θ . Now given a function $f : \Omega \rightarrow \mathbb{R}^3$ we say the function f is axially symmetric if any axially symmetric subset $S \subset \Omega$ we have that $f(S)$ is axially symmetric.

With a view to attempting to show sharpness of Theorems 1 and 3 we would like to try and construct a function that squeezes $B_1(0) \times [0, 1]$ in the center down to a point and use this to create a counter example to Theorems 1 and 3 for functions whose dilatation are not L^p for $p \geq n - 1$. We say a function $g : \Omega \rightarrow \mathbb{R}$ (where Ω is axially symmetric) is a cylindrical product function if $g(r \cos \theta, r \sin \theta, z) = p_1(r)p_2(\theta)p_3(z)$ for functions p_1, p_2, p_3 . A function $f : \Omega \rightarrow \mathbb{R}^3$ is a cylindrical product function if for each co-ordinate function is a cylindrical product function.

We will show that any axially symmetric orientation preserving cylindrical product function (whose coordinates satisfy certain monotonicity or convexity properties) that squeezes the cylinder down to a point does not have L^1 integrable dilatation.

Proposition 1. *Let $f : W^{1,1}(B_1(0) \times [0, 1]) : \mathbb{R}^3$ be a radially symmetric orientation preserving cylindrical product function, i.e. there exists functions w, v, g, h, l such that*

$$f(r \cos \theta, r \sin \theta, z) = (w(z)v(r) \cos(g(\theta)), w(z)v(r) \sin(g(\theta)), h(z)l(r)) \quad (182)$$

for some functions w, v, g, h, l . Assume each of these functions w, v, h, l are monotonic non-decreasing or non-increasing, g is non decreasing and w, h are either concave or convex. If $f(B_1(0) \times \{0\})$ consists of a single point then

$$\int_{B_1(0) \times [0, 1]} \frac{\|\nabla f\|^3}{\det(\nabla f)} dz = \infty. \quad (183)$$

Proof of Proposition 1. Suppose the proposition is false. So there exists a function f satisfying the hypotheses and

$$\int_{B_1(0) \times [0, 1]} \frac{\|\nabla f\|^3}{\det(\nabla f)} dz < \infty. \quad (184)$$

Let $u(\theta, r, z) = f(r \cos \theta, r \sin \theta, z)$, so $u(\theta, r, z) = (w(z)v(r) \cos(g(\theta)), w(z)v(r) \sin(g(\theta)), h(z)l(r))$. Since $w(0) = 0$, this function is non decreasing,

$$\frac{\partial w}{\partial z}(z) \geq 0 \text{ for a.e. } z. \quad (185)$$

Now we claim

$$\frac{\partial v}{\partial r}(r) \geq 0 \text{ for a.e. } r. \quad (186)$$

So see this we argue as follows. Define $F : B_1(0) \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ by $F(x, y, z) = (\sqrt{x^2 + y^2}, z)$. Since $JAC(\nabla F) = \det(\nabla F \nabla F^T) = 1$. Note by the Co-area formula

$$\infty > \mathcal{C} = \int_{B_1(0) \times [0, 1]} |\nabla f|^3 JAC(\nabla F) dz = \int_{[0, 1] \times [0, 1]} \int_{F^{-1}(r, z)} |\nabla f|^3 dH^1 dr dz.$$

Let $\delta \in (0, 1)$ be some small number we decide on later. We can find a set $\mathcal{I} \subset [0, \delta] \times [0, 1]$ with $|\mathcal{I}| \geq \frac{\delta}{2}$ and for any $(r, z) \in \mathcal{I}$, $\int_{F^{-1}(r, z)} |\nabla f|^3 dH^1 \leq c\delta^{-1}$. Pick $(r, z) \in \mathcal{I}$, by Holder's inequality we have

$$\int_{F^{-1}(r, z)} |\nabla f| dH^1 \leq c \left(\int_{F^{-1}(r, z)} |\nabla f|^3 dH^1 \right)^{\frac{1}{3}} \delta^{\frac{2}{3}} \leq c\delta^{\frac{1}{3}}. \quad (187)$$

How ever if v is non increasing then even for very small r we know $f(F^{-1}(r, z))$ must be the boundary of a disc with radius $o(1)$, so we must have $H^1(F^{-1}(r, z)) \sim o(1)$ which contradicts (187). This establishes (186).

Now we claim

$$\frac{\partial h}{\partial z}(z) \geq 0 \text{ for a.e. } z. \quad (188)$$

To see this first assume l is non constant, the for $r_1 \neq r_2$ we have that $l(r_1) \neq l(r_2)$. Since $f(B_1(0) \times \{0\})$ consists of a single point we must have $h(0)l(r_1) = h(0)l(r_2)$ so we must have $h(0) = 0$ and hence (188) is established.

On the other hand if l is constant then as f is orientation preserving and

$$\nabla u := \begin{pmatrix} \frac{dv}{dr} w \cos \circ g & -wv \sin \circ g \frac{dg}{d\theta} & \frac{dw}{dz} v \cos \circ g \\ \frac{dv}{dr} w \sin \circ g & wv \cos \circ g \frac{dg}{d\theta} & \frac{dw}{dz} v \sin \circ g \\ \frac{dl}{dr} h & 0 & \frac{dh}{dz} l \end{pmatrix} \quad (189)$$

and as $\det(\nabla u) = 2 \frac{\partial h}{\partial z} l \frac{\partial v}{\partial r} w^2 v \frac{\partial g}{\partial \theta} > 0$ and so by (186), (188) is established.

Now from (189) we know

$$\begin{aligned} \|\nabla u(\theta, r, z)\|_\infty^3 &\geq |\nabla u(\theta, r, z)|^3 \\ &\geq c \max \left\{ \left| w(z) v(r) \frac{dg}{d\theta}(\theta) \right|^3, \left| \frac{dv(r)}{dr} w(z) \right|^3, \left| \frac{dw(z)}{dz} v(r) \right|^3, \right. \\ &\quad \left. \left| \frac{dh}{dz}(z) l(r) \right|^3, \left| \frac{dl}{dr}(r) h(z) \right|^3 \right\}. \end{aligned} \quad (190)$$

And as by (188), (186) and the fact that g is non decreasing $\frac{dv}{dr} \geq 0$, $\frac{dh}{dz} \geq 0$ and $\frac{dg}{d\theta} \geq 0$

$$\begin{aligned} \det(\nabla u(\theta, r, z)) &= -\frac{dh}{dz}(z) l(r) \frac{dv}{dr}(r) \frac{dg}{d\theta}(\theta) v(r) w(z)^2 + \frac{dw}{dz}(z) v(r)^2 w(z) \frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) h(z) \\ &\stackrel{(185), (188)}{\leq} \frac{dw}{dz}(z) v(r)^2 w(z) \frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) h(z). \end{aligned} \quad (191)$$

So by (191), (190)

$$\left(\left(\frac{dw}{dz}(z) \right)^2 / w(z) \right) |v(r)| \left(\left| \frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) h(z) \right| \right)^{-1} \leq \frac{\|\nabla u(\theta, r, z)\|_\infty^3}{\det(\nabla u(\theta, r, z))}$$

So

$$\left(\frac{dw}{dz}(z) \right)^2 / w(z) \leq |v(r)|^{-1} \frac{\|\nabla u(\theta, r, z)\|_\infty^3}{\det(\nabla u(\theta, r, z))} \left| \frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) h(z) \right|. \quad (192)$$

Step 1. We will show that there can not exists $\delta > 0$ such that

$$\sup \left\{ \left| \frac{dw}{dz}(z) \right| : z \in (0, \delta) \right\} > \delta. \quad (193)$$

Proof of Step 1. By (192) we have that

$$\int_{[0, \delta]} w(z)^{-1} dz \leq c \int_{[0, \delta]} \frac{\|\nabla u(\theta, r, z)\|^3}{\det(\nabla u(\theta, r, z))} dz \quad (194)$$

For $(\theta, r) \in [0, 2\pi) \times [0, 1]$ define $l_{(\theta, r)} := \{(\theta, r, z) : z \in [0, 1]\}$. Let

$$\mathcal{G} := \left\{ (\theta, r) \in [0, 2\pi) \times [0, 1] : \int_{l_{(\theta, r)}} \frac{\|\nabla u(\theta, r, z)\|^3}{\det(\nabla u(\theta, r, z))} dz < \infty \right\}. \quad (195)$$

Now by Fubini $|[0, 2\pi) \times [0, 1] \setminus \mathcal{G}| = 0$. So for any $(\theta, r) \in \mathcal{G}$ by (194) we have that

$$\mu(A) := \int_A w(z)^{-1} dz$$

forms a finite measure on the interval $[0, \delta]$. Let $\epsilon \ll \delta$, now by Holder we have that

$$\begin{aligned} \log(w(\delta)) - \log(w(\epsilon)) &= \int_{\epsilon}^{\delta} \frac{d}{dz} (\log(w(z))) dz = \int_{[0, \delta]} \frac{dw}{dz}(z) / w(z) dz \\ &= \int_{[\epsilon, \delta]} \frac{dw}{dz}(z) d\mu(z) \leq \left(\int_{[\epsilon, \delta]} \left(\frac{dw}{dz}(z) \right)^2 d\mu(z) \right)^{\frac{1}{2}} \\ &= \left(\int_{[\epsilon, \delta]} \left(\frac{dw}{dz}(z) \right)^2 / w(z) dz \right)^{\frac{1}{2}} \stackrel{(192), (195)}{<} \infty. \end{aligned} \quad (196)$$

Since $w(0) = 0$ sending $\epsilon \rightarrow 0$ we have contradiction.

Proof of the Proposition completed. Firstly if w is concave, we must have $\lim_{z \rightarrow 0} w'(z) > 0$ since otherwise by the fact that $w(0) = 0$ and w is positive we would have $w \equiv 0$. So the existence of some δ satisfying (193) follows and so we have a contradiction. If w is convex and $\lim_{z \rightarrow 0} w'(z) > 0$ then again its easy to see there exists δ satisfying (193) and so have a contradiction.

So the only remain case to consider the when w is convex and $\lim_{z \rightarrow 0} w'(z) = 0$. Pick $(\theta, r) \in \mathcal{I}$, let

$$\mathcal{A} := \{z \in (0, 1) : w(z) > h(z)\} \text{ and } \mathcal{U} := (0, 1) \setminus \mathcal{A}.$$

$$\begin{aligned} \frac{\|\nabla u(\theta, r, z)\|_{\infty}^3}{\det(\nabla u(\theta, r, z))} &\stackrel{(190), (191)}{\geq} \left| \frac{dw}{dz}(z) \right|^2 v(r) / \left(w(z) \frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) h(z) \right) \\ &\geq c(\theta, r) \left| \frac{dw}{dz}(z) \right|^2 / w(z)^2 \text{ for } z \in \mathcal{A} \end{aligned} \quad (197)$$

where $c(\theta, r) := v(r) / \left(\frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) \right)$.

Now by Young's inequality for $p = 3, p' = \frac{3}{2}$ from (190) we have that

$$\begin{aligned} \|\nabla u(\theta, r, z)\|_{\infty}^3 &\geq \left(l(r)^2 \left(\frac{dh}{dz}(z) \right)^2 \right)^{p'} + \left(v(r) \frac{dw}{dz}(z) \right)^p \\ &\geq cl(r)^2 v(r) \left(\frac{dh}{dz}(z) \right)^2 \frac{dw}{dz}(z). \end{aligned}$$

So by (191) we have that

$$\frac{\|\nabla u(\theta, r, z)\|_\infty^3}{\det(\nabla u(\theta, r, z))} \geq d(\theta, r) \left| \frac{dh}{dz}(z) \right|^2 / h(z)^2 \text{ for } z \in \mathcal{U} \quad (198)$$

where $d(\theta, r) := cl(r)^2 / \left(\frac{dg}{d\theta}(\theta) \frac{dl}{dr}(r) \right) v(r)$.

Now \mathcal{A} is open so is the union of a countable collection of disjoint open intervals, denote them I_1, I_2, \dots . Thus $\mathcal{A} = \bigcup_{k=1}^\infty I_k$. We assume they have been ordered to that $\sup I_k \leq \inf I_j$ for $k > j$. Define $\alpha_{2k}, \alpha_{2k+1}$ to be the endpoints of I_k where $\alpha_{2k+1} \leq \alpha_{2k}$, i.e. $I_k = (\alpha_{2k+1}, \alpha_{2k})$. Define $J_k := (\alpha_{2k+2}, \alpha_{2k+1})$, so $[0, 1] = \bigcup_{k=1}^\infty \overline{I_k} \cup \overline{J_k}$.

Hence by Holder for any $(\theta, r) \in \mathcal{G}$ we have $\int_{[0,1]} \sqrt{\frac{\|\nabla u(\theta, r, z)\|_\infty^3}{\det(\nabla u(\theta, r, z))}} dz < \infty$. So by (195) we have

$$\begin{aligned} \infty &> \int_{\mathcal{A}} \sqrt{\frac{\|\nabla u(\theta, r, z)\|_\infty^3}{\det(\nabla u(\theta, r, z))}} \\ &\stackrel{(197)}{\geq} c(\theta, r) \int_{\mathcal{A}} \left| \frac{dw}{dz}(z) \right| / w(z) dz. \end{aligned}$$

Now as $\lim_{k \rightarrow \infty} \int_{\alpha_{2k}}^1 \frac{d}{dz} (\log(w(z))) dz = \lim_{k \rightarrow \infty} (\log(w(1)) - \log(w(\alpha_{2k}))) = \infty$ we must have

$$\lim_{k \rightarrow \infty} \int_{\mathcal{U} \cap [\alpha_{2k}, 1]} \frac{d}{dz} (\log(w(z))) dz = \infty \quad (199)$$

But note that $w(\alpha_{2k}) = h(\alpha_{2k})$ and $w(\alpha_{2k+1}) = h(\alpha_{2k+1})$ for every k . So

$$\begin{aligned} \int_{\mathcal{U} \cap [\alpha_{2k}, 1]} \frac{d}{dz} (\log(w(z))) dz &= \sum_{i=1}^k \int_{[\alpha_{2i}, \alpha_{2i-1}]} \frac{d}{dz} (\log(w(z))) dz \\ &= \int_{\mathcal{U} \cap [\alpha_{2k}, 1]} \frac{d}{dz} (\log(h(z))) dz \\ &\stackrel{(198)}{\leq} c \int_{\mathcal{U} \cap [\alpha_{2k}, 1]} \sqrt{\frac{\|\nabla u(\theta, r, z)\|_\infty^3}{\det(\nabla u(\theta, r, z))}} dz \\ &< \infty \end{aligned}$$

which contradicts (199). \square

REFERENCES

- [Ac-Fu 88] E. Acerbi; N. Fusco. An approximation lemma for $W^{1,p}$ functions. Material instabilities in continuum mechanics (Edinburgh, 1985/1986), 117 Oxford Sci. Publ., Oxford Univ. Press, New York, 1988.
- [Am-Da 90] L. Ambrosio; G. Dal Maso. A general chain rule for distributional derivatives. Proc. Amer. Math. Soc. 108 (1990), no. 3, 691-722.
- [Am-Fu-Pa 00] L. Ambrosio; N. Fusco; D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [As-Iw-Ma 10] K. Astala; T. Iwaniec; G. Martin. Deformations of annuli with smallest mean distortion. Arch. Ration. Mech. Anal. 195 (2010), no. 3, 899-971.
- [Ba 77] J.M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. 63 (1976/77), no. 4, 337-413.
- [Ba-Ja 87] J.M. Ball; R.D. James. Fine phase mixtures as minimisers of energy. Arch. Rat. Mech.-Anal, 100(1987), 13-52.
- [Ba-Ja 92] J.M. Ball, R.D. James. Proposed experimental tests of a theory of fine microstructure and the two well problem. Phil. Tans. Roy. Soc. London Ser. A 338(1992) 389-450.
- [Bo-Iw 82] B. Bojarski; T. Iwaniec. Another approach to Liouville theorem. Math. Nachr. 107 (1982), 253-262.
- [Ch-Gi-Po 07] A. Chambolle; A. Giacomini; M. Ponsiglione. Piecewise rigidity. J. Funct. Anal. 244 (2007), no. 1, 134-173.
- [Ch-Mu 03] N. Chaudhuri, S. Müller. Rigidity Estimate for Two Incompatible Wells. Calc. Var. Partial Differential Equations 19 (2004), no. 4, 379-390.

- [Cm-Co 10] M. Chermisi, S. Conti. Multiwell rigidity in nonlinear elasticity. *SIAM J. Math. Anal.* Volume 42, Issue 5, pp. 1986–2012 (2010)
- [Ch 64] Chernavskii, A.V.: Discrete and open mappings on manifolds. - *Mat. Sb.* 65, 1964, 357179 (Russian).
- [Ch 64] Chernavskii, A.V.: Continuation to “Discrete and open mappings on manifolds”. - *Mat. Sb.* 66, 1965, 471172 (Russian).
- [Ch-Ki 88] M. Chipot, D. Kinderlehrer. Equilibrium configurations of crystals. *Arch.Rat.Mech.-Anal.* 103 (1988), no. 3, 237–277.
- [Co-Sc 06] S. Conti and B. Schweizer. Rigidity and Gamma convergence for solid-solid phase transitions with $SO(2)$ -invariance. *Comm. Pure Appl. Math.* 59 (2006), no. 6, 830–868.
- [Cs-He-Ma 10] M. Csoranyi; S. Hencl; J. Maly. Homeomorphisms in the Sobolev space $W^{1,n-1}$. *J. Reine Angew. Math.* 644 (2010), 221175.
- [Ev-Ga 92] L.C. Evans; R.F. Gariepy. Measure theory and fine properties of functions. *Studies in Advanced Mathematics.* CRC Press, Boca Raton, FL, 1992.
- [Ev 10] L.C. Evan; Partial differential equations. Second edition. *Graduate Studies in Mathematics*, 19. American Mathematical Society, Providence, RI, 2010.
- [Fa-Zh 05] D. Faraco; X. Zhong, Geometric rigidity of conformal matrices. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 4 (2005), no. 4, 557–585.
- [Fed 69] H. Federer. *Geometric Measure Theory.* Springer-Verlag, 1969.
- [Fr 10] M. Fréchet. “Les dimensions d’un ensemble abstrait.” *Math. Ann.* 68, 145–168, 1910.
- [Fr-Ja-Mu 02] G. Friesecke, R. D. James and S. Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. *Comm. Pure Appl. Math.* 55 (2002), no. 11, 1461–1506.
- [Ge 62] F. W. Gehring. Rings and quasiconformal mappings in space. *Trans. Amer. Math. Soc.* 103 1962 353–393.
- [Ge-Iw 99] F.W. Gehring; T. Iwaniec. The limit of mappings with finite distortion. *Ann. Acad. Sci. Fenn. Math.* 24 (1999), no. 1, 253174.
- [Gu-Ya-Ma-Ry-Vu 05] V. Ya Gutlyanskii, V. Ya; O. Martio; V.I. Ryazanov; M. Vuorinen. Infinitesimal geometry of quasiregular mappings. *Ann. Acad. Sci. Fenn. Math.* 25 (2000), no. 1, 101–130.
- [He-Ko 93] J. Heinonen; P. Koskela. Sobolev mappings with integrable dilatations. *Arch. Rational Mech. Anal.* 125 (1993), no. 1, 8117.
- [He-Ko-Ma 06] S. Hencl; P. Koskela; J. Mal17 Regularity of the inverse of a Sobolev homeomorphism in space. *Proc. Roy. Soc. Edinburgh Sect. A* 136 (2006), no. 6, 12671785.
- [Wa 41] W. Hurewicz; H. Wallman. *Dimension Theory.* Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941
- [Iw-Ma 93] T. Iwaniec, G. Martin. Quasiregular mappings in even dimensions. *Acta Math.* 170 (1993), no. 1, 2917.
- [Iw-Sv 93] T. Iwaniec, V. Sverak. On mappings with integrable dilatation. *Proc. Amer. Math. Soc.* 118 (1993), no. 1, 181–188.
- [Je-Lor 08] R.L. Jerrard. A. Lorent. On multiwell Liouville theorems in higher dimension. <http://arxiv.org/abs/0802.0850>
- [Jo 61] F. John. Rotation and strain, *Comm. Pure Appl. Math.* 14 (1961), 391–413.
- [Ko 82] R.V. Kohn. New integral estimates for deformations in terms of their nonlinear strains. *Arch. Rational Mech. Anal.* 78 (1982), no. 2, 131172.
- [Lio 50] J. Liouville Théorème sur l’équation $dx^2 + dy^2 + dz^2 = \lambda (da^2 + d\beta^2 + d\gamma^2)$ *J. Math. Pures Appl.* 1, (15) (1850), 103.
- [Lo 05] A. Lorent. A Two Well Liouville Theorem. *ESAIM Control Optim. Calc. Var.* 11 (2005), no. 3, 310–356
- [Lo 10] A. Lorent. On functions who symmetric part of gradient are close. Preprint.
- [Ma 94] J. Manfredi; Weakly monotone functions. *J. Geom. Anal.* 4 (1994), no. 3, 393172.
- [Ma-Vi 98] J. Manfredi; E. Villamor. An extension of Reshetnyak’s theorem. *Indiana Univ. Math. J.* 47 (1998), no. 3, 11311745
- [Ma-Ri-Va 71] O. Martio; S. Rickman; J. Vaisala. Topological and metric properties of quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I* No. 488 (1971)
- [Ma 95] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge studies in advanced mathematics. 1995.
- [Mo 52] C.B. Morrey. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.* 2, (1952). 2517
- [Mu 90] S. Müller. Higher integrability of determinants and weak convergence in L^1 . *J. Reine Angew. Math.* 412 (1990), 2017.
- [Mu-Sv-Ya 99] S. Muller; V. Sverak; B. Yan. Sharp stability results for almost conformal maps in even dimensions. *J. Geom. Anal.* 9 (1999), no. 4, 671171.
- [Mu 96] S. Muller. Variational models for microstructure and phase transitions. *Calculus of variations and geometric evolution problems* (Cetraro, 1996), 85–210, *Lecture Notes in Math.*, 1713, Springer, Berlin, 1999.
- [Re 67] Yu. G. Reshetnyak. Liouville’s conformal mapping theorem under minimal regularity hypotheses. (Russian) *Sibirsk. Mat. Ž.* 8 1967 835–840.
- [Re 82] Yu. G. Reshetnyak. Stability theorems in geometry and analysis. *Mathematics and its Applications*, 304. Kluwer Academic Publishers Group, Dordrecht, 1994.

- [Ta 79] L. Tartar. Compensated compactness and applications to partial differential equations. Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, pp. 136–212, Res. Notes in Math., 39, Pitman, Boston, Mass.-London, 1979. 136-212.
- [Va 71] J. Vaisala. Lectures on n-dimensional quasiconformal mappings. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971. 144 pp.
- [Va 66] J. Vaisala. Discrete open mappings on manifolds. - Ann. Acad. Sci. Fenn. Math. 392, 1966.
- [Zh 92] K. Zhang. A construction of quasiconvex functions with linear growth at infinity. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), no. 3, 313176

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